SYMMETRY SCHEMES FOR ELEMENTARY PARTICLES

Thesis

Submitted by

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PREFACE

In this dissertation an attempt has been made to distinguish between results based on a relativistic combination of spin and SU(3) internal symmetry and one of spin and U(2) internal symmetry. An attempt has also been made to allow spin-dependent modifications to these theories so that the results are less vague in that the correct masses of particles appear in the theories.

To facilitate understanding of this subject the early part of the thesis is devoted to some aspects of group theory required in later sections. As a further introductory consideration we consider the internal group SU(3) and its non-relativistic combination with spin SU(6).

The material presented in this dissertation is asserted to be original except where explicit references are cited. The work on $\tilde{U}(8)$ and on spin dependent mass splitting has already been submitted for publication, in two papers, to Il Nuovo Cimento and Journal of Mathematical Physics.

It is a pleasure to thank Professor E.J. Squires and Dr. D.B. Fairlie for suggesting the field of investigation, and for their continuing interest in the work. Particular thanks are due to Professor Squires for many helpful discussions. The author wishes to thank Mr. G.A. Ringland for first raising his interest in symmetries, and Mr. G.F. Reid for helpful discussions.

CONTENTS

INTRODUCTION

The success of the internal symmetry group SU(3) in classifying elementary particles and as a basis for mass formulae has led many people to propose extensions to this group, with the object of combining spin and internal symmetry. The non-relativistic extension of F. Gürsey and L.A. Radicati[1] is the group SU(6). It is comparable with the group U(4) of E.P. Wigner[2], which combines spin with isospin for nuclear states.

Unlike U(4), SU(6) cannot be accepted so readily in its non-relativistic form. This is because the binding energies of particles described by SU(6) are sufficiently great to make relativistic considerations important. For U(4) binding energies are not so large.

A number of attempts have been made to produce a relativistic version of SU(6). In this work we will discuss only one of these; the group $\tilde{U}(12)$ of Delbourgo, Salam and Strathdee[3].

SU(6) and $\tilde{U}(12)$ result, of course, in more predictions than those of SU(3). We shall be concerned in showing that some of the more important of these predictions can be obtained from the analogous generalisations of the isotopic spin group SU(2) to U(4) and $\tilde{U}(8)$. In addition, we will show how to modify the theories of $\tilde{U}(12)$ and $\tilde{U}(8)$ to allow for the rather large spin-dependent deviations from the symmetry.

The material presented can be divided into three main sections. The first is a mathematical section and deals with some of the aspects of group theory required in the later sections. Unfortunately the subject is too vast to include all parts of interest. In the first half of this section we discuss, in general form, Lie groups and Lie algebras, while the second half is devoted to unitary groups and algebraic methods of finding irreducible tensors. We conclude this section with some comments on orthogonal and symplectic subgroups which will be found useful in the third section.

In the second section we discuss the relevance of SU(3) and SU(6) to elementary particle physics and derive some of the results based on these symmetries. These include the Gell-Mann - Okubo[4] mass formula and the derivation of neutron-proton magnetic moment ratio as given by Sakita[5].

The final section is concerned with the author's own work, mentioned above, in which the magnetic moments of the neutron and proton are derived and some of the difficulties concerning spin-dependent mass splitting in $\tilde{U}(12)$ and $\tilde{U}(8)$ are resolved.

CHAPTER 1

CONTINUOUS GROUPS*

Definition of a Group

A group is a set of elements G, within which there is defined an operation O with the following properties:

(i) If $g_1, g_2 \in G$ then $g \in G$ where $g = g_1 \, O \, g_2$.

(ii) $g_1 \, O(g_2 \, O \, g_3) = (g_1 \, O \, g_2) O \, g_3$.

(iii) There is an element e of G, such that

$$e \, O \, g = g \, O \, e = g.$$

(iv) For every $g \in G$ there is an element $g^{-1} \in G$, such that

$$g^{-1} \, O \, g = g \, O \, g^{-1} = e \, .$$

If in addition to the above

$g_1 \, O \, g_2 = g_2 \, O \, g_1$ the group is said to be abelian or commutative.

An example of a group is the set of all non-singular $n \times n$ matrices, which is a group with respect to matrix multiplication.

Many groups have a matrix representation. That is, to every element g of the group G we can associate a matrix M, such that if M_1 and M_2 correspond to g_1 and g_2 then

$$M = M_1 \times M_2$$

corresponds to

$$g = g_1 \, O \, g_2 \, .$$

Since both $M_1 \times M_2$ and $M_2 \times M_1$ belong to the representation, the matrices M are clearly square matrices and of the same dimension.

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* A large part of the material of this chapter has been discussed by C. Fronsdal (6) but the works of Weyl (7) and Hammermesh (8) have also been consulted.

We shall be concerned only with groups which have a matrix representation. We may suppose the group to be defined by a given matrix representation. We further restrict our attention to groups whose elements are continuous functions of a finite number of parameters, α_i say. By this we mean the matrix elements are continuous in the parameters. To every set of values α_i we associate a matrix or group element. Nothing is lost by the assumption that $\alpha_i = 0$ corresponds to the identity matrix I. In the neighbourhood of the identity we expect an expansion

$$M = I + \alpha_i L_i + \alpha_i \alpha_j L_{ij} + \dots \tag{1.1}$$

Let us write

$$M = I + \varepsilon L \tag{1.2}$$

where ε is small and fixed.

Then to second order in ε

$$M^{-1} = I - \varepsilon L + \varepsilon^2 L^2$$

Considering the commutator $C = M_A M_B M_A^{-1} M_B^{-1}$ of two infinitesimal matrices M_A and M_B, we have to second order in ε

$$C = 1 + \varepsilon^2 [L_A, L_B] \tag{1.3}$$

where $M_A = 1 + \epsilon L_A$: $M_B = 1 + \epsilon L_B$.

Thus since C is a member of the set of matrices M $\varepsilon [L_A, L_B]$ is a member of the matrices L .

Since L_A and L_B and $\varepsilon [L_A, L_B]$ are essentially linear combinations of the matrices L_i of equation (1.1), then by taking

L_A and L_B to be multiples of L_i and L_j we have

$$[L_i, L_j] = C_{ij}^k L_k \tag{1.4}$$

The numbers C_{ij}^k are called structure constants. They satisfy the relations

$$C_{ij}^k + C_{ji}^k = 0 \tag{1.5}$$

and

$$C_{ij}^\ell \; C_{\ell k}^m + C_{ki}^\ell \; C_{\ell j}^m + C_{jk}^\ell \; C_{\ell i}^m = 0 \tag{1.6}$$

(1.5) follows from (1.4), while (1.6) is a consequence of the Jacobi identity

$$[L_i, [L_j, L_k]] + [L_k, [L_i, L_j]] + [L_j, [L_k, L_i]] = 0 \tag{1.7}$$

Any set of matrices with the property (1.4) is a basis for a Lie algebra. A Lie algebra is a set of matrices such that all complex multiples and linear combinations belong to the set and such that the commutator of any two of the matrices belong to the set.

If a second set of matrices L_i should satisfy (1.4) with the same structure constants, we say that the two associated Lie-algebras have the same structure.

If a set of numbers C_{ij}^k satisfy the relations (1.5) and (1.6), then we can find a set of matrices L_i forming the basis of a Lie-algebra. Thus defining $(L_i)_j^{\;k} = - C_{ij}^k$ and rewriting (1.6)

$$-(-C_i)_j^{\;\ell}(-C_k)_\ell^{\;m} - C_{ki}^\ell(-C_\ell)_j^{\;m} + (-C_k)_j^{\;\ell}(-C_i)_\ell^{\;m} = 0$$

we have $\quad [L_k, L_i]_j^{\ m} = C_{ki}^{\ \ell}(L_\ell)_j^{\ m}$

The choice $(L_i)_j^{\ k} = -C_{ij}^{\ k}$ is called the fundamental representation of the algebra.

The reason why we are so interested in infinitesimal transformations is that once these are known the finite transformations can be built up, or at least those finite transformations continuously connected to the identity. To see this suppose we wish to construct the finite element M corresponding to $\alpha_i = c_i$, say. In α_i-space join the point $\alpha_i = 0$ to $\alpha_i = c_i$ by a curve $\alpha_i = \alpha_i(t)$ with $\alpha_i(0) = 0$ and $\alpha_i(1) = C_i$. Denoting $M(\alpha_i(t))$ by $M(t)$ so that $M(0) = I$ and $M(1) = M$, we have for small δt

$$M(t + \delta t)\, M^{-1}(t) = I + L(t)\delta t \quad ,$$

or

$$M(t + \delta t) = (I + L(t)\delta t)\, M(t)$$

Thus

$$\begin{aligned} M = M(1) &= (I + L(1)\delta t) M(1 - \delta t) \\ &= \prod_{t=0}^{1} (I + L(t)\delta t) \end{aligned} \qquad (1.8)$$

A group may contain matrices M which are not continuously connected to the identity. In these circumstances we have a set of disjoint continuous cosets and we have to supplement our knowledge of the group in the neighbourhood of the identity by a discrete set of matrices (one for each coset) in order to reconstruct the finite group.

In order to classify groups we introduce the following definitions concerning Lie algebras.

(a) A sub-algebra is a subset of the matrices L_A which is itself an algebra.

(b) A sub-algebra is said to be invariant if the commutator of one of its matrices L_A with an arbitrary matrix of the algebra belongs to the sub-algebra.

(c) An algebra is said to be abelian if its structure constants vanish.

(d) A semi-simple algebra has no abelian invariant sub-algebra.

(e) A simple algebra has no invariant sub-algebra.

In the following, we will consider mostly simple Lie algebras, but before we investigate these it is useful to consider the following connection between a Lie algebra and a linear vector space. This will give us a geometrical picture of the algebra.

The basis matrices L_i of a Lie algebra are analogous to vectors, forming a basis of a vector space; the general vector of the space corresponding to a linear combination of the L_A with complex coefficients. We may further extend the analogy by comparing the commutator of two elements of the algebra with a generalised vector product of the two corresponding vectors.

If L and L' satisfy $[L, L'] = 0$ they are said to be orthogonal.

We now consider an important result which will enable us to construct semi-simple Lie algebras once the simple ones are known.

Let L_A, L_α' form the basis of an algebra with an invariant sub-algebra given by L_A. Then

$$[L_A, L'_\alpha] = C^B_{A\alpha} L_B$$

$$\text{and} \quad [L_A, L_B] = C^C_{AB} L_C \tag{1.9}$$

$$\text{and} \quad [L'_\alpha, L'_\beta] = C^A_{\alpha\beta} L_A + C^\gamma_{\alpha\beta} L'_\gamma$$

We would like to replace L'_α by a set L_α such that $C^A_{\alpha\beta} = 0$ and $C^B_{A\alpha} = 0$. We would then have two invariant sub-algebras L_A and L_α with

$$[L_A, L_\alpha] = 0 \tag{1.10}$$

Thus a semi-simple algebra would be formed simply by adding together the algebras of simple Lie algebras and imposing the constraint, (1.10).

The equations (1.9) imply

$$\begin{aligned} C^\beta_{A\alpha} &= 0 = C^\beta_{\alpha A} \\ C^\gamma_{AB} &= 0 \end{aligned} \tag{1.11}$$

$$\sum_{\substack{\alpha\beta\gamma \\ \text{cyclic}}} \left\{ C^E_{\alpha\beta} C^\rho_{E\gamma} + C^\varepsilon_{\alpha\beta} C^\rho_{\varepsilon\gamma} \right\} = 0$$

but since by (1.11) $C^\rho_{E\gamma} = 0$, this is

$$\sum_{\substack{\alpha\beta\gamma \\ \text{cyclic}}} C^\varepsilon_{\alpha\beta} C^\rho_{\varepsilon\gamma} = 0 \tag{1.12}$$

For the fundamental representation in which $(L_\alpha)^\gamma_\beta = - C^\gamma_{\alpha\beta}$

L_α form a sub-algebra.

It follows from (1.9) that $C^A_{\alpha\beta} = 0$.

Our second requirement $C^B_{A\alpha} = 0$ may depend on a careful choice of L_α.

Suppose we put

$$L_\alpha = L'_\alpha + \mu_\alpha^D L_D \tag{1.13}$$

Then

$$[L_\alpha, L_A] = (C^B_{\alpha A} + \mu_\alpha^D C^B_{DA}) L_B$$

and since L_B are independent

$[L_\alpha, L_A] = 0$ requires

$$(C_\alpha + \mu_\alpha^D C_D)^B_A = 0 \tag{1.14}$$

We have shown that, under the assumption of the existence of invariant subgroup, $C^\beta_{A\alpha} = C^\gamma_{AB} = C^A_{\alpha\beta} = 0$

It is clear from the fundamental representation that the elements of the group can be represented by matrices of the form

$$\left(\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right) \tag{1.15}$$

A group with elements of this form is said to be reducible. If we can also satisfy (1.14), we may then also have $Y = 0$, in which case the group is said to be completely reducible. We shall consider only groups of this type. We note above that a change of basis of the Lie algebra $L_A \to L'_A$ results in a new set of structure constants $C^C_{AB} \to C'^C_{AB}$, e.g. Equation ~~(1.18)~~ ~~(13)~~ (1.13)

Structure constants of different algebras which are related in this way are said to be equivalent. Between the elements ~~of~~ g and g' of the corresponding groups there is a relation of the form

$$g' = T g T^{-1} \tag{1.16}$$

where T is a constant matrix. Under a transformation for which T is a fixed element of the group the $C_B{}^A{}_C$ are unchanged.

We now investigate simple Lie algebras.

Let ℓ be the maximum number of mutually commuting generators that can be formed from the n generators L_A of a group. Then ℓ is called the rank of the group or algebra. Let us suppose that the commuting generators are: $L_1 = H_1, \ldots\ldots, L_\ell = H_\ell$ and denote the remaining generators by E_α. To distinguish suffixes referring to the commuting and non-commuting generators we shall introduce Roman and Greek suffixes, thus: H_i, E_α. When we wish not to specify or sum over both we shall use capital letters.

We have

$$[H_i, \; H_j] = 0 \tag{1.17}$$

It can be shown that H_i and the related $(C_i)^B_A$ can be chosen diagonal. With this simplification

$$[H_i, \; E_\alpha] = C_{i\alpha}^{\;A} L_A = C_{i\alpha}^{\;\alpha} E_\alpha \tag{1.18}$$

(α not summed).

Introducing $$r_i(\alpha) = C_{i\alpha}^{\alpha} \tag{1.19}$$

(1.18) becomes

$$[H_i, E_\alpha] = r_i(\alpha) E_\alpha \tag{1.20}$$

Equations (1.18) and (1.20) give us a special form of C_{AB}^{C} with

$$C_{ij}^{A} = C_{i\alpha}^{j} = 0 \quad \text{and} \quad C_{i\alpha}^{\beta} = r_i(\alpha)\delta_\alpha^{\beta} \tag{1.21}$$

The form of the remaining structure constants is obtained by considering the Jacobi identy

$$\sum_{\text{cyclic}} [H_i, [E_\alpha, E_\beta]] = 0$$

This gives

$$\left(C_{iA}^{B} C_{\alpha\beta}^{A} - C_{\alpha\beta}^{B} r_i(\beta) - C_{\alpha\beta}^{B} r_i(\alpha)\right) L_B = 0$$

But $(C_i)_A^B$ is diagonal.

We have therefore

$$C_{\alpha\beta}^{B}\left(C_{iB}^{B} - r_i(\alpha) - r_i(\beta)\right) = 0$$

putting $B = j, \gamma$ in turn

$$C_{\alpha\beta}^{j}\left(r_i(\alpha) + r_i(\beta)\right) = 0 \tag{1.22}$$

$$C_{\alpha\beta}^{\gamma}\left(r_i(\gamma) - r_i(\alpha) - r_i(\beta)\right) = 0 \tag{1.23}$$

Introducing

$$\underline{\alpha} = (r_1(\alpha), r_2(\alpha), \text{-----}, r_\ell(\alpha))$$

we have

$$c_{\alpha\beta}^{j} = 0 \qquad \text{unless} \qquad \underline{\alpha} + \underline{\beta} = \underline{0} \tag{1.24}$$

and

$$c_{\alpha\beta}^{\gamma} = 0 \qquad \text{unless} \qquad \underline{\gamma} = \underline{\alpha} + \underline{\beta} \tag{1.25}$$

$\underline{\alpha}$ is called a root-vector.

We may thus introduce $r_i(\alpha)$, $N_{\alpha\beta}$ such that

$$c_{\alpha\beta}^{i} = r^i(\alpha) \qquad \text{when} \qquad \underline{\alpha} + \underline{\beta} = 0$$

and

$$c_{\alpha\beta}^{\gamma} = N_{\alpha\beta} \qquad \text{when} \qquad \underline{\gamma} = \underline{\alpha} + \underline{\beta}$$

We may, with a careful allocation of suffixes, write

$$[E_\alpha, E_{-\alpha}] = r_i^i(\alpha) H_i \tag{1.26}$$

and

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \tag{1.27}$$

Some ~~cases~~ care, however, must be taken as $\alpha + \beta = \gamma$.
may have more than one solution α, β; that is to say, we could have an E_7 and an E_7', say,

with $[E_2, E_5] = N_{25} E_7$

and $[E_3, E_4] = N_{34} E_7'$.

In such cases the various E_γ's may be distinguished by dashes etc. The need for such a distinction can be avoided. For suppose there are s E's; call one E_1 and a second E_s; commute $[E_1, E_s] = N_{1s} E_{1+s}$
and $[E_1, E_{1+s}]$, $[E_s, E_{1+s}]$ etc.
until no new E's arise. Also form $[E_{-1}, E_s]$ and $[E_1, E_{-s}]$ etc.

There will be no ambiguity. If all E's are not now accounted for choose another and call it E_{s^2} and repeat the process above. Ultimately introduce E_{s^3} , E_{s^4} ,....... until all E's are exhausted. There will be no ambiguity of suffixes. In practice, however, it will usually be more convenient to suffer some ambiguity and use small integer suffixes.

We may simplify the form of C^C_{AB} yet further.

We introduce the matrix

$$g_{ij} = \sum_\alpha r_i(\alpha)\, r_j(\alpha) \tag{1.28}$$

This is a submatrix of the matrix

$$g_{AB} = C^E_{AD}\, C^D_{BE} \tag{1.29}$$

The other non-zero elements form a second submatrix

$$g_{\alpha\beta} = \left\{ \sum_i C^\alpha_{\alpha i}\, C^i_{-\alpha\alpha} + \sum_\gamma C^{\alpha+\gamma}_{\alpha\gamma}\, C^\gamma_{-\alpha,\alpha+\gamma} \right\} \delta_{\alpha,-\beta}$$

$$= \left\{ \sum_i r_i(\alpha)\, r^i(\alpha) + \sum_\gamma N_{\alpha\gamma}\, N_{-\alpha,\alpha+\gamma} \right\} \delta_{\alpha,-\beta} \tag{1.30}$$

This suggests that we try to bring about the normalisation

$$g_{ij} = \delta_{ij} \tag{1.31}$$

$$g_{\alpha\beta} = \begin{cases} 1 & \text{if } \beta = -\alpha \\ 0 & \text{otherwise.} \end{cases} \tag{1.32}$$

To achieve this let us first consider the effect of replacing E_α by $d_\alpha E_\alpha$. By considering the commutation relations (1.20), (1.26) and (1.27) we see that this results in the changes

$$
\begin{aligned}
r^i(\alpha) &\longrightarrow d_\alpha d_{-\alpha}\, r^i(\alpha) \\
N_{\alpha\beta} &\longrightarrow d_\alpha d_\beta / d_{\alpha+\beta}\, N_{\alpha\beta}
\end{aligned}
\tag{1.33}
$$

Combining this with (1.30)

$$
g_{\alpha,-\alpha} \longrightarrow d_\alpha d_{-\alpha}\, g_{\alpha,-\alpha} \tag{1.34}
$$

We choose the product $d_\alpha d_{-\alpha}$ to bring about the normalisation (1.32).

Having brought about this normalisation we have the result

$$
r_i(\alpha) = g_{ij}\, r^j(\alpha) \tag{1.35}
$$

This previously read

$$
g_{\alpha,-\alpha}\, r_i(\alpha) = g_{ij}\, r^j(\alpha) \tag{1.36}
$$

Equation (1.35) gives us another reason for desiring the normalisation (1.31). To achieve this we replace H_i by $V_i{}^j H_j$. This brings about the transformations

$$
r_i(\alpha) \longrightarrow V_i{}^j\, r_j(\alpha)
$$

and

$$
r^i(\alpha) \longrightarrow r^j(\alpha)\, (V^{-1})_j{}^i \tag{1.37}
$$

From (1.28) and (1.30) we see

$$
g_{ij} \longrightarrow V_i{}^l V_j{}^m g_{lm} \tag{1.38}
$$

and

$$
g_{\alpha,-\alpha} \longrightarrow g_{\alpha,-\alpha}
$$

With a suitable matrix $V = (V_i{}^j)$ we can bring about the result (1.31).

We now have the simple description

$$[H_i\,,\,H_j] = 0$$
$$[H_i\,,\,E_\alpha] = r_i(\alpha)\,E_\alpha$$
$$[E_\alpha\,,\,E_{-\alpha}] = \sum_i r^i(\alpha)\,H_i \qquad (1.39)$$
$$[E_\alpha\,,\,E_\beta] = N_{\alpha\beta}\,E_{\alpha+\beta}$$

wi th

$$r^i(\alpha) = r_i(\alpha)$$

and

$$\sum_\alpha r_i(\alpha)\,r_j(\alpha) = \delta_{ij} \qquad (1.40)$$

The special form we have chosen for our structure constants leads to a very simple geometrical picture of the group in the vector space of the root vectors $\underline{\alpha}$.

Consider the identity

$$\sum_{cyclic} [E_\alpha, [E_\beta, E_{-\beta}]] = 0$$

This gives

$$-\underline{\alpha}\cdot\underline{\beta} + N_{-\beta,\alpha+\beta}\,N_{\alpha\beta} + N_{\beta,\alpha-\beta}\,N_{-\beta\,\alpha} = 0$$

or

$$N_{\alpha\beta}\,N_{-\beta,\alpha+\beta} = N_{\alpha-\beta,\beta}\,N_{-\beta\,\alpha} + \underline{\alpha}\cdot\underline{\beta} \qquad (1.41)$$

If the generators $E_{\alpha+\beta}$ and $E_{\alpha-\beta}$ vanish then $\underline{\alpha}\cdot\underline{\beta} = 0$.

We can obtain a series of generators

$$E_\alpha\ ,\ E_{\alpha+\beta}\ ,\ E_{\alpha+2\beta}\ ,\ \dots \tag{1.42}$$

by considering $[E_\alpha, E_\beta] = N_{\alpha\beta}\, E_{\alpha+\beta}$,

$$[E_{\alpha+\beta}, E_\beta] = N_{\alpha+\beta,\beta}\, E_{\alpha+2\beta}, \dots\dots \qquad \text{etc.}$$

Putting $\alpha + q\beta$ for α in (1.41) and writing

$$\nu_q = N_{\alpha+q\beta,\beta}\, N_{-\beta,\alpha+(q+1)\beta} \tag{1.43}$$

we have

$$\nu_q = \nu_{q-1} + \beta\cdot(\alpha + q\beta) \tag{1.44}$$

The series (1.42) and the series

$$E_\alpha\ ,\ E_{\alpha-\beta}\ ,\ E_{\alpha-2\beta}\ ,\ \dots$$

obtained by replacing E_β by $E_{-\beta}$ in the above must terminate at some stage due to the vanishing of $N_{\alpha+q\beta,\beta}$ for some q.

Let $\nu_Q = \nu_{P-1} = 0$, these vanishing due to the vanishing of $N_{\alpha+Q\beta,\beta}$ and $N_{\alpha-(P-1)\beta,\beta}$.

By iteration of (1.44) we have

$$\nu_q = \sum_{p=-P}^{q} \beta\cdot(\alpha + p\beta)$$

$$= \sum_{s=0}^{P+q} \beta\cdot\{(\alpha - P\beta) + s\beta\}$$

$$= \beta\cdot(\alpha - P\beta)(P+q+1) + \tfrac{1}{2}(P+q)(P+q+1)\,\beta\cdot\beta \tag{1.45}$$

and since $N_Q = 0$

We have

$$\underline{\beta} \cdot (\underline{\alpha} - P\underline{\beta}) + \tfrac{1}{2}(P+Q)\underline{\beta}\cdot\underline{\beta} = 0 \qquad (1.46)$$

whence $Q - P = \dfrac{-2\underline{\alpha}\cdot\underline{\beta}}{\underline{\beta}\cdot\underline{\beta}}$ is an integer, and since $E_{\alpha + (Q-P)\beta}$ exists

$$\underline{\gamma} = \underline{\alpha} - \left(\frac{2\,\underline{\alpha}\cdot\underline{\beta}}{\underline{\beta}\cdot\underline{\beta}}\right)\underline{\beta} \qquad (1.47)$$

is a root.

If $\underline{\alpha} + \underline{\beta} \neq 0$ then $\underline{\alpha}$ and $\underline{\beta}$ are independent.

For put

$$\underline{\beta} = V\underline{\alpha}$$

then

$$\frac{2}{V} \text{ and } 2V \text{ are integers}$$

from the above. Hence $V = \pm 1, \pm 2$

but since $[E_\alpha, E_\alpha] = 0$, 2 and similarly -2 is ruled out.

± 1 are of course allowed.

From

$$\sum_{\text{cyclic}} [E_\alpha, [E_\beta, E_\gamma]] \qquad \text{with } \underline{\alpha} + \underline{\beta} + \underline{\gamma} = 0$$

we deduce

$$N_{\alpha\beta}\,\underline{\gamma} + N_{\beta\gamma}\,\underline{\alpha} + N_{\gamma\alpha}\,\underline{\beta}$$

and since any two of $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ are independent

$$N_{\alpha\beta} = N_{\beta, -\alpha-\beta} = N_{-\alpha-\beta, \alpha} \qquad (1.48)$$

Thus

$$\mathcal{N} = N_{\alpha\beta}\, N_{-\beta, \alpha+\beta} = N_{\alpha\beta}\, N_{-\alpha, -\beta} \qquad (1.49)$$

Previously we have considered the transformation

$$E_\alpha \longrightarrow d_\alpha E_\alpha$$

which gives $\quad N_{\alpha\beta} \longrightarrow (d_\alpha d_\beta / d_{\alpha+\beta}) N$

and we have fixed the value of the products $d_\alpha d_\beta$, but the ratio d_α/d_β is still at our disposal.

We note that

$$N_{\alpha\beta}/N_{-\alpha,-\beta} \longrightarrow d_\alpha/d_{-\alpha} \cdot d_\beta/d_{-\beta} \cdot d_{-\alpha-\beta}/d_{\alpha+\beta} \cdot N_{\alpha\beta}/N_{-\alpha,-\beta}$$

the ratio $d_\alpha/d_{-\alpha}$ may be chosen so that

$$N_{\alpha\beta} = -N_{-\alpha,-\beta} \tag{1.50}$$

Thus $\quad \mathcal{N} = -(N_{\alpha\beta})^2$

but from (1.45) and (1.46)

$$\mathcal{N} = -\tfrac{1}{2}(P+1)(P+Q)\underline{\beta}\cdot\underline{\beta} + \tfrac{1}{2}P(P+1)\underline{\beta}\cdot\underline{\beta}$$

$$= -\tfrac{1}{2}(P+1)\,Q\,\underline{\beta}\cdot\underline{\beta}$$

Hence

$$N_{\alpha\beta} = \pm\sqrt{(P+1)Q/2}\;|\underline{\beta}| \tag{1.51}$$

The sign $\pm$ is to some extent at our disposal as the signs of d_α may be changed without violating (1.49) and (1.50).

We now show an even simpler relation for the angles between root vectors. For if $\phi_{\alpha\beta}$ is the angle between $\underline{\alpha}$ and $\underline{\beta}$

then $$\cos^2\phi_{\alpha\beta} = \frac{(\underline{\alpha}\cdot\underline{\beta})^2}{(\underline{\alpha}\cdot\underline{\alpha})(\underline{\beta}\cdot\underline{\beta})}$$

$$= \frac{1}{4}\left(\frac{2\underline{\alpha}\cdot\underline{\beta}}{\underline{\alpha}\cdot\underline{\alpha}}\right)\left(\frac{2\underline{\alpha}\cdot\underline{\beta}}{\underline{\beta}\cdot\underline{\beta}}\right) = \frac{1}{4} \text{ x integer}$$

Hence

$$\phi_{\alpha\beta} = 90^\circ, 60^\circ, 45^\circ, 30^\circ$$
$$120^\circ, 135^\circ\ 150^\circ .$$

In order that both

$$\left(\frac{2\underline{\alpha}\cdot\underline{\beta}}{\underline{\alpha}\cdot\underline{\alpha}}\right) \quad \text{and} \quad \left(\frac{2\underline{\alpha}\cdot\underline{\beta}}{\underline{\beta}\cdot\underline{\beta}}\right) \quad \text{be}$$

integers we require

$$\frac{|\underline{\alpha}|}{|\underline{\beta}|} = 1 \quad \text{when} \quad \phi_{\alpha\beta} = 60^\circ \text{ or } 120^\circ$$

$$\frac{|\underline{\alpha}|}{|\underline{\beta}|} = \frac{1}{\sqrt{2}} \text{ or } \sqrt{2} \quad \text{when} \quad \phi_{\alpha\beta} = 45^\circ \text{ or } 135^\circ$$

and

$$\frac{|\underline{\alpha}|}{|\underline{\beta}|} = \frac{1}{\sqrt{3}} \text{ or } \sqrt{3} \quad \text{when} \quad \phi_{\alpha\beta} = 30^\circ \text{ or } 150^\circ .$$

If we can find a set of vectors $\underline{\alpha}$ satisfying the above and equation (1.40), we will with the aid of equation (1.51) be able to construct the structure constants of a group. Equation (1.47) will assist us in forming such a set of vectors. Geometrically (1.47) says that if $\underline{\alpha}$ and $\underline{\beta}$ are two roots then the vector $\underline{\gamma}$ (in the plane of $\underline{\alpha}$ and $\underline{\beta}$) which is the reflection of $\underline{\alpha}$ in the hyperplane perpendicular to $\underline{\beta}$) is also a root.

We could set about the problem of finding all the groups of a given rank ($\ell \geqslant 2$) in the following way.

The problem of finding the structure constants, for all simple Lie algebras of rank ℓ, is thus equivalent to introducing a set of vectors in ℓ-dimensional space such that the ratios of lengths and angles are in agreement with the above prescription and such that if $\underline{\alpha}$ and $\underline{\beta}$ are two such members then so is $\underline{\gamma}$ defined by (1.47). When this is done we will have

$$\sum_{\alpha} r_i(\alpha)\, r_j(\alpha) \propto \delta_{ij}$$

so we will have to introduce a normalisation constant at this stage. With the aid of (1.50) and (1.51) we can then construct $N_{\alpha\beta}$; the values of P and Q being determined by consideration of the existence of root vectors $\underline{\alpha} + q\underline{\beta}$.

To classify the groups we need then consider only the configuration of the un-normalised root-vectors.

It turns out that there are four infinite strings of groups corresponding to the simple Lie algebras (i.e. they are simple generalisations of one another) and five exceptional ones. They are

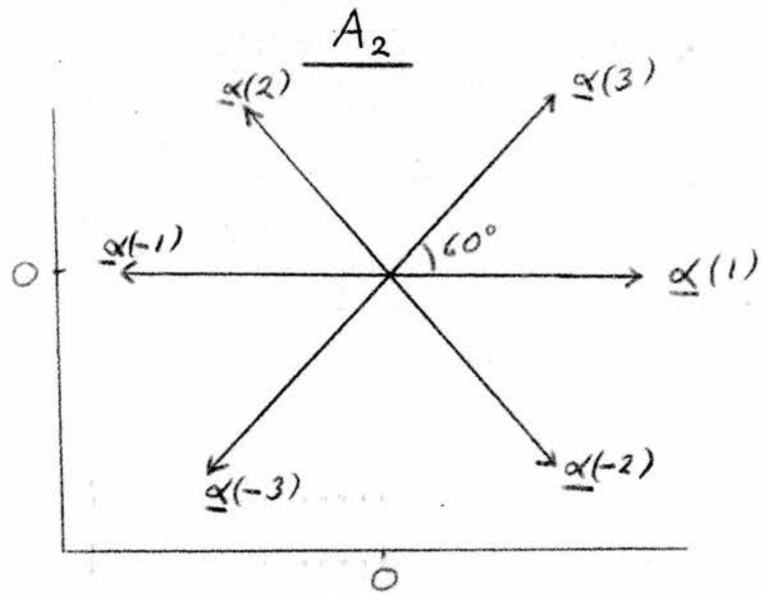
A_ℓ ; $\ell = 1,2,\ldots.$ of order $\ell^2 + 2\ell$,
B_ℓ ; $\ell = 2,3,\ldots.$ of order $2\ell^2 + \ell$,
C_ℓ ; $\ell = 3,4,\ldots.$ of order $2\ell^2 + \ell$,
D_ℓ ; $\ell = 4,5,\ldots.$ of order $2\ell^2 - \ell$

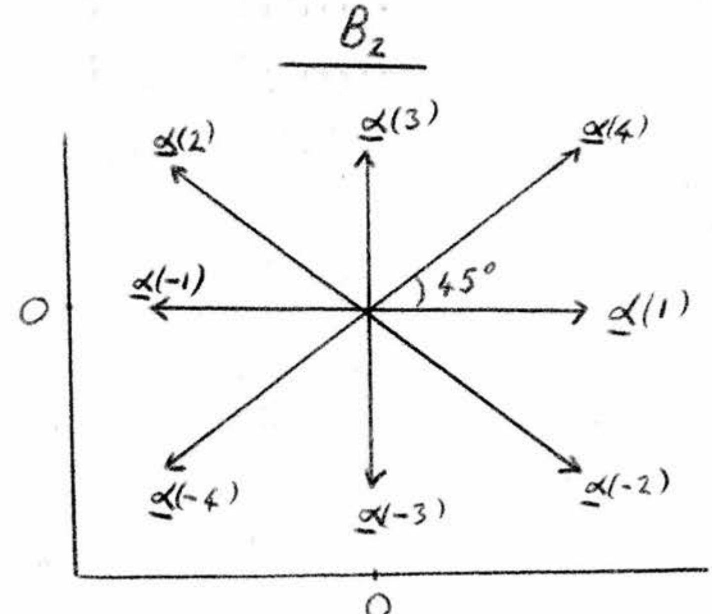
and $G_2(14)$, $F_4(52)$, $E_6(78)$, $E_7(133)$, $E_8(248)$.

This classification is due to Cartan. A_ℓ is otherwise known as $SU_{\ell+1}$ and will be discussed in the next chapter. The root vectors of A_ℓ make only angles of 60° or 90° with one another. B_ℓ is also known as $O_{2\ell+1}$ the orthogonal group in

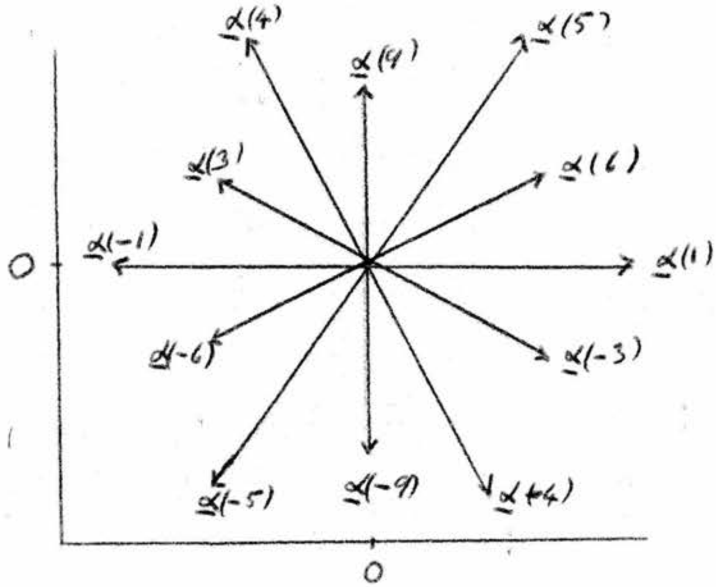
$2\ell+1$ dimensions. B_1 is omitted from the list as it is the same as A_1. C_ℓ are also known as $Sp(2\ell)$, the simplectic group in 2ℓ dimensional space. ($Sp(2\ell)$ is the group which leaves invariant an antisymmetric form in 2ℓ-dimensional space). Finally D_ℓ is otherwise known as $O_{2\ell}$. We have omitted C_2 as it is the same as B_2 and C_1 as it is the same as A_1 and B_1. D_1, D_2 and D_3 are omitted because D_1 and D_2 are abelian and D_3 happens to be the same as A_3. D_2 is also not simple ; $D_2 \equiv A_1 \times A_1$.

The root-vectors of the rank two groups can be represented by the following root diagrams:





G_2



$\underline{G}_2$ is the only group with root vectors at angles of 30° with one another. This is evident from the fact that if a group has two root vectors at an angle of 30° or 150° then by use of the result (1.47) it contains a set of vectors with the configuration of those of G_2. It is evident however that the only way we can have more vectors is that they be perpendicular to the plane of this configuration. This results in a non-simple group however (we have the direct product of G_2 and some other group). In general we must avoid configurations in which the vectors can be separated into two sets which span orthogonal spaces.

Now that we know how to construct the structure constants of a group we are left with the problem of finding the representations of the group. We will be more interested in representations of the generators of the group.

To do this we use the procedure employed for the spin-group SU(2). The generators here are usually denoted iJ_z, iJ_+, iJ_- rather than H, E_{+1}, E_{-1}. The method here is to introduce state vectors $\psi(j,m)$ with the properties

$$J^2\psi(j,m) = j(j+1)\psi(j,m)$$
$$J_z\psi(j,m) = m\psi(j,m)$$

where

$$J^2 = J_z^2 + J_+J_- + J_-J_+$$

is an invariant for the representation.

By considering the commutator relations

$$[J_z, J_\pm] = \pm J_\pm$$
$$[J_+, J_-] = J_z$$

We discover

$$J_+\psi(j,m) \propto \psi(j,m+1)$$
$$J_-\psi(j,m) \propto \psi(j,m-1)$$

By requiring the set $\psi(j,m)$ to be finite in number we are forced to choose $2j$ an integer and m to take the values $j, j-1, j-2, \ldots, 1-j, -j$.

The procedure for other groups is similar, but more complicated. For the group with generators $H_1, \ldots H_\ell$, $E_{\pm\alpha_1}$, $E_{\pm\alpha_2}, \ldots, E_{\pm\alpha_s}$, we introduce vectors $\psi(c_1, \ldots c_k, m_1, \ldots, m_\ell)$ These are eigenvectors of $H_1, \ldots, H_\ell$ with eigenvalues $m_1, \ldots, m_\ell$, and of certain other operators $C_1, \ldots, C_k$ with eigenvalues $c_1, \ldots, c_k$. The operators $C_1, \ldots, C_r$ are of two kinds. The first kind is analogous to J^2 of SU(2) and simply label the representation of the group considered. The second type are not invariant; they are usually invariants for a subgroup.

The operator E_{α_i} has a more complicated effect than J_+.

Not only does it change the value of m_i, but it also causes changes in the values of the second kind of C_i. It may further not lead to just one new basis vector, but a linear combination of several.

We will not treat this problem here, but will consider the special case of SU(3) in Chapter 3.

CHAPTER 2

UNITARY GROUPS*

The special unitary group of n-dimensions (by special we mean of determinant +1) lends itself to simple algebraic analysis. The lowest dimensional representation is the defining group in n-dimensions. All higher representations correspond to the transformations of various tensor fields formed from the basic 'spinor field'. We will concentrate on the irreducible tensor fields as these are more easily handled and suffice to obtain the important results.

In order to understand our procedure let us look at an example; the rotation group in 3 dimensions. The basic tensor is the 3 component vector x_i. This transforms under the orthogonal transformation a_{ij}, thus

$$x_i' = a_{ij} x_j \tag{2.1}$$

We can introduce the notion of a cartesian tensor $s_{ij\ldots k}$ with the transformation law

$$s'_{ij\ldots k} = a_{ii'} a_{jj'} \ldots a_{kk'} s_{i'j'\ldots k'} \tag{2.2}$$

$s_{ij\ldots k}$ may be thought of as a component of a vector in 3r dimensional space where r is the number of suffixes $ij\ldots k$ or rank of the tensor s. It transforms under the generalised transformation $a_{ij,\ldots k\,;\,i'j'\ldots k'}$ thus

$$S'_{ij\ldots k} = a_{ij\ldots k;\,i'j'\ldots k'} S_{i'j'\ldots k'} \tag{2.3}$$

* A more extensive treatment of Unitary Groups has been given by by Hammermesh (8). References (7) and (8) have been heavily consulted in preparing this chapter.

where $$a_{ij\ldots\ldots k;\, i'j'\ldots\ldots k'} = a_{ii'}\, a_{jj'} \ldots\ldots a_{kk'} \tag{2.4}$$

The matrices $a_{ij\ldots.k;\; i'j'\ldots k'}$ constitute a representation of the rotation group. This is evident, since there is a 1 - 1 correspondence between the tensors with components a_{ij} and the generalised tensors with components $a_{ij\ldots k;\; i'j'\ldots k'}$, which is preserved under their respective multiplication laws.

The 3r dimensional representation will, however, not in general be irreducible. We illustrate this by showing that the second rank tensor $T_{\alpha\beta}$ separates into three invariant subspaces; by this we mean that under the most general transformation a_{ij} of the rotation group, the components of the new subspaces are expressed as linear combinations of the components of the corresponding subspace.

We first separate out the trace T of $T_{\alpha\beta}$. Thus

$$T_{\alpha\beta} = \frac{1}{3} T \delta_{\alpha\beta} + T^{(T)}_{\alpha\beta} \quad . \tag{2.5}$$

where $T_{\alpha\beta}$ is traceless.

Second we can separate $T^{(T)}_{\alpha\beta}$ into its symmetric and anti-symmetric parts.

Thus

$$T_{\alpha\beta} = \frac{1}{3} T \delta_{\alpha\beta} + S_{\alpha\beta} + A_{\alpha\beta} \tag{2.6}$$

where

$$S_{\alpha\beta} = \frac{1}{2}\left\{ T^{(T)}_{\alpha\beta} + T^{(T)}_{\beta\alpha} \right\}$$

and $$\tag{2.7}$$

$$A_{\alpha\beta} = \frac{1}{2}\left\{ T^{(T)}_{\alpha\beta} - T^{(T)}_{\beta\alpha} \right\}$$

We have decomposed the 9-component tensor $T_{\alpha\beta}$ into a tensor with only one independent component (namely $T\,\delta_{\alpha\beta}$); the traceless symmetric tensor $S_{\alpha\beta}$) and one with 3 independent components (the antisymmetric tensor $A_{\alpha\beta}$).

That this decomposition is invariant rests on the fact that the operations of taking the trace, symmetrizing and antisymmetrizing commute with the rotation matrices a_{ij}.

Thus

$$
\begin{aligned}
S'_{\alpha\beta} &= \tfrac{1}{2}\left\{ T'^{(T)}_{\alpha\beta} + T'^{(T)}_{\beta\alpha} \right\} \\
&= \tfrac{1}{2}\left\{ a_{\alpha\alpha'} a_{\beta\beta'} T^{(T)}_{\alpha'\beta'} + a_{\beta\beta'} a_{\alpha\alpha'} T^{(T)}_{\beta'\alpha'} \right\} \\
&= a_{\alpha\alpha'} a_{\beta\beta'} \tfrac{1}{2}\left\{ T^{(T)}_{\alpha'\beta'} + T^{(T)}_{\beta'\alpha'} \right\} \\
&= a_{\alpha\alpha'} a_{\beta\beta'} S_{\alpha'\beta'}
\end{aligned}
$$

We now have the result that the direct product of the 3-dimensional rotation-matrix with itself leads to a 9 component representation of the rotation group which decomposes according to the scheme

$$3 \times 3 \quad = \quad 5 + 3 + 1 \tag{2.8}$$

Let us now return to the group SU(n). Here the basis spinor has n components. We can clearly introduce the notion of Cartesian tensors for this group and in analogy with the above , an r-th rank tensor will form a basis for an r-dimensional representation of SU(n) which will once again be reducible. For SU(n) however only the operations of symmetry and antisymmetry are at our disposal for reducing tensor spaces. The operation of taking the trace depends on the relation

$$a_{ii'}\, a_{jj'}\, \delta_{i'j'} = \delta_{ij}$$

$$\text{or} \qquad A\,A^T = I$$

in matrix notation; which is the definition of the orthogonal matrix. Thus for SU(3) we have the equation

$$3 \times 3 = 6 + \bar{3} \tag{2.9}$$

in place of (2.8). The significance of the bar over the 3 will be explained later.

To establish the result that only symmetrization processes are required to construct irreducible tensors from the cartesian tensors, we first consider the general group of transformations in n-dimensionsl space GL(n). That is the group of all n x n matrices with complex elements. Clearly only symmetrization is available for the reduction of tensors of this group. Tensors of subgroups of GL(n) may be further reducible than those of GL(n) since subgroups are obtained from GL(n) by placing restrictions on the matrix elements.

Suppose that under a subgroup H of GL(n) an irreducible tensor formed from a cartesian tensor of rank r of GL(n) is reducible under H. This means that by taking linear combinations of the tensor space we can arrange the invariant subspaces of H in sequence. A transformation of H will thus be in the form of a sequence of square matrix blocks situated along the diagonal of a larger matrix, all other elements being zero. Now the elements of this transformation are homogeneous polynomials of degree r in the n^2 elements of transformation of the basic spinor. The vanishing of certain elements in the larger matrix imply conditions on the n^2 elements, the conditions imposed by restricting the group to H.

We will now show that certain subgroups of GL(n) do not lead to any further reduction of the irreducible terms of GL(n). Consider the general group of real transformations in n-dimensional space GR(n). Now suppose that a certain irreducible representation of GL(n) is reducibel under GR(n). From the above discussion we see that this implies that certain homogeneous polynomials in the matrix elements of the basic n x n transformation of GL(n) vanish under GR(n). But this is just the restriction that they be real. We know, however, that if a polynomial vanishes for all real values of its variables it vanishes for all complex values also, and so the polynomials are zero for GL(n) also. Thus reducibility for GR(n) implies reducibility for GL(n) and hence an irreducible representation of GL(n) is also irreducible for GR(n).

The group U(n), the group of all unitary matrices of n-dimensions, leads to no further reduction of the irreducible representations of GL(n). To show this we consider the generators of U(n). These are the set of all hermitian matrices and may be chosen to be

$i > j$ $\quad X_{ij}$: matrix with 1 in ij and ji position, but otherwise zero.

$\quad X^{ii}$: matrix with 1 in ii position, but otherwise zero.

$i > j$ $\quad Y^{ij}$: matrix with i in ij position, -i in ji position but otherwise sero.

An infinitesimal unitary matrix U is given by

$$U = 1 + iS$$

where S is a linear combination of X^{ij} and Y^{ij} with real infinitesimal parameters. If these parameters were allowed also complex values we would be back with GL(n). Suppose we have an irreducible representation of GL(n), then this will be irreducible for U(n) also, for suppose it were reducible for U(n) denote this representation by $\tilde{U}$, then $\tilde{U}$ and hence $\tilde{S}$ can be cast into the form of a sequence of blocks about the diagonal of $\tilde{U}$ or $\tilde{S}$ with all other matrix elements zero.

$$\tilde{U} = 1 + i\tilde{S} \quad .$$

But since each real parameter of $\tilde{S}$ may vary independently this means each generator $\tilde{X}^{ij}$, $\tilde{Y}^{ij}$ of $\tilde{U}$ is in block form and will of course remain so even for complex multiples. Thus GL(n) would also be reducible. Hence the irreducible representations of GL(n) are irreducible under U(n) also. If we restrict the above groups by requiring that the matrices of the group be of determinant +1, no further reduction is brought about. This is easily established for U(n) since U(n) = U(1) x SU(n) and since all representations of U(1) are one dimensional, the dimension of representation of U(n) and of the corresponding representation of SU(n) are the same. This argument applies also to GL(n) = U(1)xU(1)x SL(n) and GR(n) = U(1) x SR(n).

From these results the one of interest to us is that every irreducible representation of GL(n) remains irreducible under the group SU(n). This establishes the result that only symmetrization processes are at our disposal for the reduction of tensors with cartesian suffixes for the Group SU(n).

We have now to present those symmetrization processes which allow no further reduction by further symmetrization.

Two such processes immediately spring to mind, the processes of complete symmetrization and antisymmetrization. Thus suppose we have a tensor of rank f, that is f indices, then we may form from it two tensors of definite symmetry by the operators

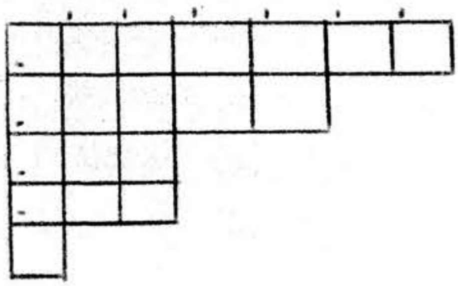
$$\underline{s} = \sum_{p} \underline{p} \tag{2.10}$$

and

$$\underline{a} = \sum_{q} \delta_q \underline{q} \tag{2.11}$$

where $\underline{p}$ and $\underline{q}$ are permutation operators acting on the indices of the tensor and δ_q is the sign of the permutation $\underline{q}$.

To generalise these operations we partition the f indices of the tensor into a number of sets f_i, thus $f_1 + f_2 + \dots = f$, and arrange these in rows under one another in decreasing order of length. We can imagine these to be in f boxes forming a pattern. Thus



(2.12)

is the pattern $(f_1, f_2, f_3, f_4, f_5) = (7, 5, 3, 3, 1)$ for a tensor of rank $f = 19$.

The symmetry operator associated with this pattern is the operator

$$\underline{c} = \sum_{p,q} \delta_q \underline{q}\,\underline{p} \tag{2.13}$$

where $\underline{p}$ is a permutation interchanging suffixes so that they remain in the same row and $\underline{q}$ a permutation leaving suffices in the same column. $\underline{c}$ may also be written

$$\underline{C} = \left(\sum_{q} \delta_q \underline{q}\right)\left(\sum_{p} \underline{p}\right). \tag{2.14}$$

To form a tensor of symmetry type $(f_1, f_2, \ldots)$ from the general tensor of rank $f = f_1 + f_2 + \ldots$ we first arrange the suffixes in the form of the pattern $(f_1, f_2, \ldots)$; we then form the partially symmetrized tensor by symmetrizing with respect to the rows and finally we antisymmetrize this tensor with respect to the columns.

The operators $\underline{a}$ and $\underline{s}$ of equations (2.10) and (2.11) satisfy the relations

$$\underline{a} \cdot \underline{a} = f!\, \underline{a} \tag{2.15}$$

$$\underline{s} \cdot \underline{s} = f!\, \underline{s} \tag{2.16}$$

Thus if $\underline{a}$ or $\underline{s}$ is applied to a tensor which is already antisymmetrized or symmetrized, the result is simply a multiple of the original tensor. We will clearly require $\underline{C}$ to have this property if it is to be useful to us. That is, we require

$$\underline{C} \cdot \underline{C} = \gamma\, \underline{C} \tag{2.17}$$

From equation (2.14) we have

$$\begin{aligned} \underline{C} \cdot \underline{p} &= \underline{C} \\ \underline{q} \cdot \underline{C} &= \delta_q\, \underline{C} \end{aligned} \tag{2.18}$$

where $\underline{p}$ and $\underline{q}$ are any row and column permutations.

(2.14) may also be written

$$\underline{C} = \sum_{s} C(s)\, \underline{S} \tag{2.19}$$

with $C(S) = \delta_q$ when $\underline{s} = \underline{q} \cdot \underline{p}$ and $C(s) = 0$ when $\underline{s}$ is not of this form. Note if $\underline{s} = \underline{q} \cdot \underline{p}$, $\underline{q}$ and $\underline{p}$ are unique, i.e. if $\underline{s} = \underline{q}' \cdot \underline{p}'$ also then $\underline{q}' = \underline{q}$ and $\underline{p}' = \underline{p}$.

(2.18) and (2.19) imply

$$C(sp) = C(s)$$
$$C(qs) = \delta_q C(s) \tag{2.20}$$

Suppose we have a function $a(s)$ with the properties

$$a(sp) = a(s)$$
$$a(qs) = \delta_q a(s) \tag{2.21}$$

Then $a(s)$ is a multiple of $c(s)$, for if $\underline{s} = \underline{q}.\underline{p}$, then from (2.21)

$$a(q\ p) = \delta_q\ a(1)$$

Putting $a(1) = \lambda$, we clearly have

$a(s) = \lambda\ C(s)$ whenever $\underline{s} = \underline{q}\ \underline{p}$. If s does not have this form there exist transpositions $\underline{U}$ and $\underline{V}$ belonging to p and q respectively such that $\underline{SU} = \underline{VS}$. (2.22)

Hence

$$a(s) = a(su) = a(vs) = -a(s) \tag{2.23}$$

and hence

$$a(s) = \lambda\ C(s) \tag{2.24}$$

in general.

The above proof of this result hangs rather critically on the assertion (2.22).

We restate the proposition. A permutation $\underline{s}$ is of the form $\underline{qp}$ if and only if two suffixes originally in the same row are not sent into the same column.

The condition is clearly necessary. We have to show that an $\underline{s}$ not satisfying (2.22) may be factorised in the form $\underline{q}.\underline{p}$. Consider the suffixes in the first row, of length f_1. After the application of S these must occupy f_1 different columns (or two will be in the same column). They can be moved to the final column by a permutation p. We note that the overhang of f_1 over f_2 already has suffixes in their final positions by this move. It follows that the suffixes of the second row must occupy a position in each of the first f_2 rows and similarly for other rows. We can thus bring each suffix into its final column by a set of disjoint row permutations (one for each row). There remains only an operation of the type (q) to arrive at $\underline{s}$, which proves the assertion.

We are now in a position to establish the result (2.17).

If we substitute (2.19) into (2.17) we have

$$\sum_{s,t} C(s)\,C(t)\,\underline{s}\cdot\underline{t} = \gamma \sum_{s} C(s)\,\underline{s}$$

or

$$\sum_{t} C(st)\,C(t^{-1}) = \gamma\, C(s)$$

We write this symbolically

$$C.C = \gamma C$$

Consider $C.\chi.C$, $\chi = \chi(s)$. This clearly has the properties of a(s) of equation (2,21) and is thus a multiple of C. In particular we have

$$C.C = \gamma C \tag{2.25}$$

Considering the special case $s = 1$

$$\sum_{t} C(t)\,C(t^{-1}) = \gamma$$

We state the result

$$\gamma g = f! \tag{2.26}$$

where g is the dimensionality of the subgroup of the symmetric group Π_f, satisfying $\underline{c}.\underline{s}$ ~~be~~longs the subgroup.

Put $\underline{e} = \underline{c}/\gamma$ then $\underline{e}.\underline{s}$ generate an irreducible subgroup of Π_f. For suppose $\underline{e} = \underline{e}_1 + \underline{e}_2$, then with $\underline{e}_1 . \underline{e}_1 = \underline{e}_1$ etc.

Then

$$\underline{e} . \underline{e}_1 . \underline{e} = \underline{e}\,\underline{e}_1 = \underline{e}_1$$

but $$\underline{e} . \underline{e}_1 . \underline{e} = \lambda \underline{e}$$

hence $$\underline{e}_1 = \lambda \underline{e}$$

and $$\underline{e}_1 . \underline{e}_1 = \lambda^2 \underline{e} = \underline{e}_1$$

i.e. $$\underline{e}_1 = \underline{e} \quad \text{or} \quad \underline{e}_1 = 0$$

The pattern P' is said to be higher than P if the first non-vanishing number in the series $f_1' - f_1$, $f_2' - f_2$,..... is positive.

If P' is higher than P then

$$C' . C = 0 \tag{2.27}$$

To prove this result we make use of the result that two suffixes in the same row of P' are in the same ~~row~~ column of P. Let $f_k' - f_k$ be the first non-vanishing term in the series above. Then the redistribution of suffixes in the first k - 1 rows of P' when placing them in P, must completely fill the overhang of f_k or else the result is already true. The redistribution of the f_k' suffixes has however then only f_k columns in which to be placed so the result is true in this event too.

Let the transposition of two such suffixes be v

then $$C'(s\,v) = C'(s)$$

and $$C(v\,s) = -C(s)$$

$$c'.c(s) = \sum_t c'(st^{-1})c(t)$$

Writing $v\,t$ in place of t

$$c'.c(s) = \sum_t c'(st^{-1}v)\,c(vt)$$
$$= -\sum_t c'(st^{-1})c(t) = -c'.c(s)$$

which proves the result.

There are many important results which can be proved for the symmetric group π_f. The proofs demand a knowledge of group characters, which we do not discuss here. We state such results as we need, or are of particular interest.

Different patterns P correspond to inequivalent subgroups of π_f. (That is, are not isomorphic).

Tensors of rank f of SU(n) with $f_n = 0$ of symmetry type P and P' are inequivalent if P and P' are different. (That is, they transform differently even if the numbers of independent components happens to be the same).

The restriction to patterns with $f_n = 0$ arises from the fact that if we antisymmetrize more than n numbers, only n of which are different, the result is zero and, since our suffixes take on only n different values, tensors with more than n rows are zero, (Such tensors are antisymmetric in the first column of suffixes). If the first column has just n suffixes, we may factor it off as the totally antisymmetric tensor in n dimensions. This tensor remains form invariant under SU(n). Under U(n), however, it changes by a factor equal to the determinant of the transformation. The restriction $f_n = 0$ is not required for U(n). For SU(n) then, the set of n - 1 numbers $f_1 - f_2$, $f_2 - f_3, \ldots\ldots, f_{n-1} - f_n$ specify the type of symmetry.

In placing the suffixes of a tensor in the form of a pattern P for the purposes of symmetrization we make some choice in the order in which they are arranged. Let us call the suffixes $i_1, i_2, \ldots, i_f$; then the natural order or book order is $i_1, \ldots, i_{f_1}$ in the first row, $i_{f_1+1}, \ldots, i_{f_2}$ in the second row etc. Let us suppose this is the order we have chosen for the operator $\underline{C}$. Suppose we had chosen another order in which to place the suffixes, say a permutation $\underline{r}$ of the natural order. Then the symmetry operator for this arrangement $\underline{Cr}$ is related to $\underline{C}$ by the equation

$$\underline{Cr}.\underline{r} = \underline{r}.\underline{C} \tag{2.28}$$

or

$$Cr(s) = C(r^{-1}sr) \tag{2.29}$$

Clearly $\underline{Cr}$ projects out from Π_f a subgroup isomorphic to that projected out by $\underline{C}$, however $\underline{Cr}$ will in general differ from $\underline{C}$. We may expect then that the general tensor of rank f will give rise to more than one tensor of a given symmetry type.

Let us consider the general tensor of rank 3, $T_{i_1 i_2 i_3}$ From this we can form a symmetric tensor;

$$S_{i_1 i_2 i_3} = (\underline{S}\ T)_{i_1 i_2 i_3} \tag{2.30}$$

an antisymmetric tensor;

$$A_{i_1 i_2 i_3} = (\underline{a}\ T)_{i_1 i_2 i_3} \tag{2.31}$$

and two independent tensors;

$$Y^{(1)}_{i_1 i_2 i_3} = (\underline{Y}^{(1)}\ T)_{i_1 i_2 i_3} \tag{2.32}$$

$$Y^{(2)}_{\substack{i_1 \, i_3 \\ i_2}} = (\underline{Y}^{(2)} T)_{i_1 i_2 i_3} \tag{2.33}$$

where $\underline{S}$ and $\underline{a}$ are symmetrizer and antisymmetrizer for Π_3 and $\underline{Y}^{(1)}$ and $\underline{Y}^{(2)}$ are $\underline{C}$ operators for the suffixes in the order i_1, i_2, i_3 and i_1, i_3, i_2 respectively. Explicitly

$$\underline{Y}^{(1)} = (\underline{I} - (i_1, i_3))(\underline{I} + (i_1 i_2)) \tag{2.34}$$

$$\underline{Y}^{(2)} = (\underline{I} - (i_1 i_2))(\underline{I} + (i_1, i_3)) \tag{2.35}$$

Where $\underline{I}$ is the identity permutation and (i_1, i_2) denotes a transposition operator.

We observe

$$\underline{Y}^{(1)} \cdot \underline{Y}^{(2)} = \underline{Y}^{(2)} \cdot \underline{Y}^{(1)} = 0 \tag{2.36}$$

The identity operator $\underline{I}$ may be expanded

$$\underline{I} = \frac{1}{6}\underline{S} + \frac{1}{6}\underline{a} + \frac{1}{3}\underline{Y}^{(1)} + \frac{1}{3}\underline{Y}^{(2)} \tag{2.37}$$

Thus

$$T_{i_1 i_2 i_3} = \frac{1}{6} S_{i_1 i_2 i_3} + \frac{1}{6} A_{i_1 i_2 i_3} + \frac{1}{3} Y^{(1)}_{\substack{i_1 \, i_2 \\ i_3}} + \frac{1}{3} Y^{(2)}_{\substack{i_1 \, i_3 \\ i_2}} \tag{2.38}$$

There are now two questions we wish to answer. The first is, how many independent components are there belonging to a tensor of SU(n) with definite symmetry. The second is, suppose we multiply two tensors of given symmetry together in an outer product so that we now have a reducible tensor, what is the

the symmetry of the irreducible tensors which result from the decomposition of the product. To answer the second question it is convenient to write in terms of symmetry patterns, e.g.

$$\yng(1) \times \yng(1) = \yng(2) + \yng(1,1) \tag{2.39}$$

or

$$\yng(2) \times \yng(1) = \yng(3) + \yng(2,1) \tag{2.40}$$

Equation (2.39) is obvious since we can clearly symmetrize or antisymmetrize with respect to the indices of the two tensors and nothing else. Equation (2.40) follows from the fact that the first tensor being symmetrical in its two indices must keep these in the same row.

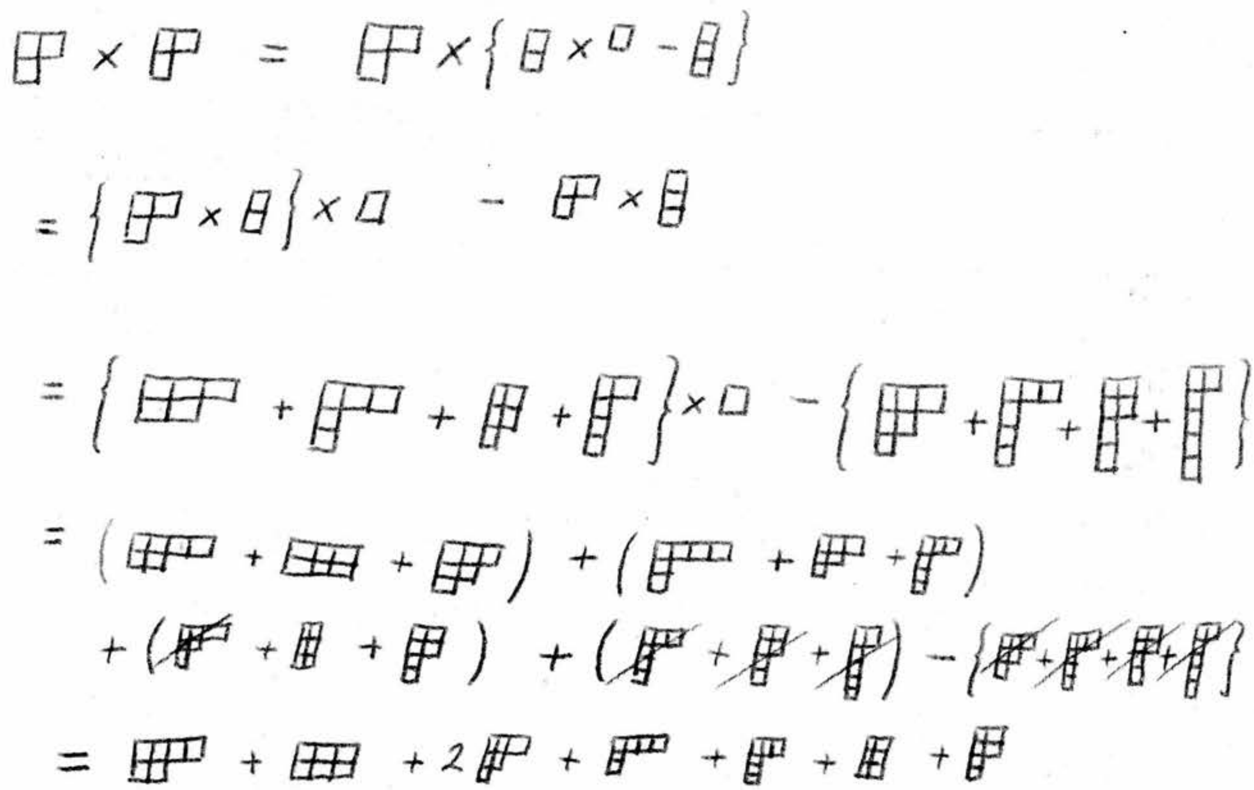
More generally we may multiply two patterns together by adding the squares of the second to the first in all possible ways which are consistent with some rules. One general rule is that the result should be a pattern. We consider the two special cases in which the second pattern is either a column or a row.

If the second pattern is a column we must add the squares to the first pattern so that no two are placed in the same row.

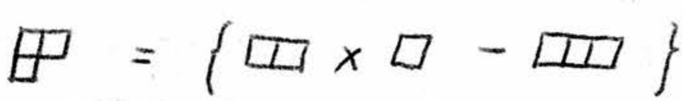
If the second pattern is a row we must add the squares so that no two fall in the same column.

Since we are essentially dealing with the algebra of symmetry operators, it is clear that associative and distributive laws hold for this multiplication.

The method of dealing with more complicated multiplication is illustrated by the following example.



In working out this example we considered patterns with as many as 5 rows. This is of course necessary for SU(n) with n $\geqslant$ 5 but for SU(3), for example, we could have safely rejected all diagrams with more than 3 rows throughout as tensors with such symmetry vanish identically for SU(3). An alternative method of tackling this example would have been to expand the second pattern



and proceed similarly.

Any pattern can be broken down into sums and differences of products of column patterns. This is evident, since if we consider the product of a pattern differing from that required by absence of the right-most column with a column tensor of that length, we can obtain an expression for the required tensor in terms of this product and tensors with fewer boxes in the final column. By repeated application of this procedure applied to the by-products we finally have only products of tensors with fewer columns. These in turn can be repeatedly reduced until we have finally only products of column tensors.

The number of independent components of a row or column tensor is easily calculated, so we could use the above decomposition to calculate the dimension of any given tensor.

The dimension of a row tensor:
Consider a row tensor of SU(n) with f indices $i_1, \ldots, i_f$. In order to specify the tensor we need only specify those components with $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_f \leqslant n$ or

$$0 < i_1 < i_2 + 1 < i_3 + 2 < \ldots < i_f + f-1 < n+f \quad ,$$

all other components being obtainable from one of these by a permutation.

The answer is

$$\frac{(n+f-1)(n+f-2)\ldots\ldots(n+1)n}{f!}$$

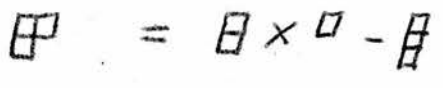
For a column tensor with $f \leqslant n$ components we need specify only those components with

$$0 < i_1 < i_2 < \ldots < i_n \leq n$$

i.e. $$\frac{n(n-1)\ldots\ldots(n-f+1)}{f!}$$

components.

Although the above procedure may prove useful in simple cases, e.g. from



with $n \geqslant 3$ we have

$$\text{Dim. } \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;=\; \frac{n(n-1)}{2} \times n \;-\; \frac{n(n-1)(n-2)}{6}$$

$$= \frac{n(n^2-1)}{3}$$

in general it is much easier to use the formula

$$\text{Dim}\,\{f_1, \ldots, f_n\} = \frac{D(h_1, \ldots\ldots, h_n)}{D(n-1, n-2, \ldots, 1, 0)} \tag{2.41}$$

where

$$D(a_1, \ldots\ldots, a_n) = \prod_{i<j} (a_i - a_j) \tag{2.42}$$

and

$$h_i = f_i + n - i \tag{2.43}$$

We omit the derivation of this formula.

We now allow our tensors, indices which transform under transformations

$$a^+ = a^{-1} \tag{2.44}$$

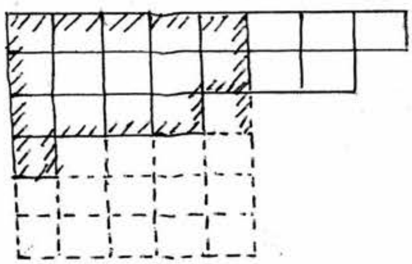
while other indices transform under a. To distinguish the two kinds of indices we write those transforming under a^+ as upper indices and those transforming under a as lower indices. It is convenient to write a and a^+ each with one upper and one lower index, in such a way that when transforming a tensor summations occur between an upper and lower index.

We note

$$(a^+)_i{}^j \, a_j{}^k = \delta_i{}^k \tag{2.45}$$

So we now have an invariant tensor $\delta_i{}^k$. This will alow us to separate out traces from tensors with upper and lower indices. This operation is however equivalent to a symmetrization process of an equivalent tensor with only lower indices. As we will now show, the introduction of the upper index has only provided us with an alternative way of representing tensors of a given symmetry. To see this, we note the existence of the invariant tensor $\varepsilon^{i_1 - - - - i_n}$ of SU(n).

If we have a tensor of definite symmetry with only lower indices we may multiply it with $\varepsilon^{i_1 - - - - - - - i_n}$ without loss; we may further contract a number of the indices of $\varepsilon^{i_1 - - - i_n}$ with those of a single column of the tensor since such indices are antisymmetric and nothing is lost, that is the number of independent components remains the same. A useful way of using this freedom is to contract every index in the first row with an index in $\varepsilon^{i_1 - - - - - - - i_n}$. If the length of the column is f_1^* we will now have a tensor with $n - f_1^*$ upper indices. We may now go on to remove the second and third columns of the tensor replacing them with columns of upper indices of lengths $n - f_2^*$ and $n - f_3^*$ respectively. The tensor we now have will have two patterns associated with it, one for its upper indices and one for its lower. The tensor will have the symmetry of these patterns in the respective indices. To indicate the distribution of indices in the upper-index pattern, we consider the diagram of SU(6) below.



This represents a tensor which originally had 8 columns. The first 5 columns have been removed as described above. The indices i_5 and i_6 now fill the two dotted squares in the first column;

i_6 in the lower square. Similarly a set i_4' i_5' i_6' fills the dotted completion of the second column.

In this way we could entirely remove the pattern of lower indices and have just upper indices. The resulting pattern of upper indices is called the adjoint pattern. It is the pattern

$$(f_1 - f_n,\ f_1 - f_{n-1},\ ----\ ,\ f_1 - f_2,\ 0) \tag{2.46}$$

It is often convenient to work with tensors with both upper and lower indices - an economy in the number of indices is one advantage.

For the group SU(3) we introduce the following simple description; in place of the pattern f_1, f_2, f_3 we can, by the removal of the first f_2 columns, have two row patterns $f_1 - f_2$, 0, 0 and $f_2 - f_3$, 0, 0 for lower and upper indices respectively. The general SU(3) tensor will thus be equivalent to a tensor which is completely symmetric in its upper and lower indices and traceless with respect to contraction between any upper and lower index.

The Subgroups O(n) and Sp(n) of SU(n).

If we restrict the transformation matrices $a_j^{\ i}$ of SU(n) by the requirement

$$\sum_k a_i^{\ k} a_j^{\ k} = \delta_{ij} \tag{2.47}$$

we no longer have the group SU(n), but the subgroup O(n). Under O(n) the components δ_{ij} are invariant, and so constitute a tensor. Combining (2.47) with (2.45)

$$a_i^{\ j} = a_i^{\ j\,*} \tag{2.48}$$

so $a_i^{\ j}$ are real.

The invariant tensor δ_{ij}, and its inverse δ^{ij} may be used to lower and raise indices. The irreducible tensors of SU(n) which we have obtained by symmetrisation processes can now be further reduced by the operation of trace contraction. Thus from the tensor $T_{i_1 i_2 i_3 \cdots\cdots i_f}$ of rank f we can form a tensor of rank f-2

$$T^{(12)}_{i_3 \cdots\cdots i_f} = \delta^{i_1 i_2} T_{i_1 i_2 i_3 \cdots\cdots i_f} \tag{2.49}$$

In order that $T_{i_1 \cdots\cdots\cdots i_f}$ be irreducible under O(n) we require that it be traceless with respect to all pairs of indices and of definite symmetry type. It is of course possible to obtain irreducible tensors of O(n) by other requirements but these then are essentially equivalent to the above prescription for lower rank tensors. It turns out for tensors whose symmetry pattern has the sum of its first two columns greater than n, that the above prescription gives the zero tensor. If we are simply interested in O(n) and not concerned with the fact that it is a subgroup of SU(n), it is simpler to contract out traces first and then to symmetrize. In this way we associate a pattern with a given irreducible tensor of O(n). We may, however, wish to answer the question; what irreducible tensors under O(n) are contained in a given irreducible tensor of SU(n). Under these circumstances symmetrization is already taken care of so we have to subtract out traces at this stage.

The group Sp(n) is the group of n x n matrices with the property that an antidymmetric matrix C_{ij} with non-vanishing determinant should remain invariant. That is, if $a_i^{\ j} \varepsilon Sp(n)$

$$c_{ij} = a_i^{\ i'} a_j^{\ j'} c_{i'j'} \tag{2.50}$$

Equation (2.50) closely resembles

$$\delta_{ij} = a_i^{\ i'} a_j^{\ j'} c_{i'j'}$$

~~of~~ which it is true for O(n).

In analogy with O(n) we can form from a tensor of rank f, one of rank $f-2$, by using C^{ij} the inverse of C_{ij} to contract. Thus from $T_{i_1 \cdots\cdots i_f}$ we form

$$T^{(12)}_{i_3 \cdots\cdots i_f} = C^{i_1 i_2} T_{i_1 i_2 i_3 \cdots\cdots i_f} \tag{2.51}$$

Just as for O(n), we can decompose the general tensor of rank f into invariant subspaces, by means of trace operations. To do this we express $T_{i_1 \cdots\cdots i_f}$ as the sum of a tensor traceless with respect to every pair of indices and a remainder of the form

$$\sum_{\alpha\beta} \frac{1}{n} C_{i_\alpha i_\beta} T_{i_1 \cdots i_{\alpha-1} i_{\alpha+1} \cdots i_{\beta-1} i_{\beta+1} \cdots i_f} \tag{2.52}$$

The remainder (2.52) can be decomposed by subtracting out all double traces. By repeated application of this process and following this up with symmetrization we arrive at irreducible tensors of Sp(n). The analysis for O(n) runs parallel.

We note that Sp(n) is defined only for even n, as an antisymmetric matrix is necessarily singular for odd n. It turns out that symmetry patterns with more than n/2 rows lead to zero tensors when 'traceless' Cartesian tensors are symmetrized.

CHAPTER 3

SU(3) AND ELEMENTARY PARTICLES

The idea of internal symmetry is to group together sets of elementary particles with similar properties and regard them as different states of the same particle. The different states of the particle are distinguished by internal quantum numbers which are conveniently chosen to coincide with the eigenvalues of the state labelling operators of the group.

For example the neutron and proton are often thought of as states of the nucleon. The symmetry group involved here is SU(2) or isotopic spin. In analogy with ordinary spin where states are often labelled $|j, j_3\rangle$, SU(2) states are labelled $|i, i_3\rangle$. The neutron and proton form a doublet and belong to the two dimensional representation. We make the correspondence

$$\begin{aligned} |p\rangle &= |\tfrac{1}{2}, \tfrac{1}{2}\rangle \\ |n\rangle &= |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{aligned} \tag{3.1}$$

The quantum numbers of SU(2), however, are not sufficient to specify uniquely all particles. Gell-Mann and Nishijima[9] have, with the aid of the extra additive quantum numbers Strangeness S and Baryon number B, classified the strongly interacting particles. Thus the particles are classified into representations of U(1) x U(1) x SU(2). Having classified particles in this way, we can construct theories which are invariant under this group. The problem is similar to constructing spin-independent theories.

Theories invariant under the group will have multiplets with identical properties. That we have multiplets at all will only be evident through the statistics of particles.

However, in order to be realistic, such theories have to be supplemented by symmetry breaking modifications. Thus the neutron and proton are distinguished by their interaction with the electromagnetic field. This situation is comparable with an electron placed in a uniform magnetic field; the anisotropy of space in this case distinguishing states with spin parallel and antiparallel to the field. For unitary space, however, the 'anisotropy' is natural and permanent, so that a special direction is singled out for reference.

An extension to the group SU(2) was proposed by Sakata[10], who, noticing that p, n and Λ are of roughly the same mass, suggested that they belong to the 3-dimensional representation of SU(3); p and n thus maintaining their invariance under SU(2) which is now a subgroup of SU(3). The additive quantum number B is still required in the classification of particles, but S is replaced by the hypercharge quantum number Y which is an eigenvalue of the generator Y of SU(3), which commutes with the SU(2) subgroup.

We do not classify particles according to the Sakata model.

The defining representation of SU(3) is the set of matrices e^{iH}, where H is any hermitian traceless 3 x 3 matrix. We may expand it in the form

$$H = \sum_{i=1}^{8} \alpha_i F_i \qquad (3.2)$$

where F_i are the basis matrices of Gell-Mann[11]. We give them below together with the notation of de Swart[12].

$$F_1 = I_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad F_2 = I_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_3 = I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad F_4 = K_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$F_5 = K_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad F_6 = L_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.3}$$

$$F_7 = L_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad F_8 = M = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$K_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad L_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The matrices have been chosen such that

$$\mathrm{Tr.}\ F_i^{\,2} = \frac{1}{2} \tag{3.4}$$

The I's and K's and L's generate SU(2) subgroups.

It is sometimes convenient to work with the matrices [12]

$$(A_i^{\,j})_{\alpha\beta} = \delta_{j\alpha}\,\delta_{i\beta} - \frac{1}{3}\,\delta_{ij}\,\delta_{\alpha\beta} \tag{3.5}$$

In analogy with (3, 2) we have

$$H = \sum_{ij} \beta_i^{\,j} A_i^{\,j}$$

but now we must put

$$\beta_i^{\,j} = \beta_j^{\,i\,*} \tag{3.6}$$

as $A_i{}^j = (A^+)^i_j$. We also have one redundant parameter, say $\beta_3{}^3$ as $A_1{}^1 + A_2{}^2 + A_3{}^3 = 0$.

In terms of the operators A^j_i , F_i we can define two higher order operators

$$F^2 = \sum_{i=1}^{8} F_i^2 = \frac{1}{2} A_i^j A_j^i \tag{3.7}$$

and

$$G^3 = \frac{1}{2}\left(A_i^k A_j^i A_k^j + A_k^i A_i^j A_j^k\right) \tag{3.8}$$

These are called Casimir operators. They commute with all generators of the group and will serve to label the representations. A general state vector will thus be expressible in terms of the basis vectors

$$| g, f ; i, i_3, y > \tag{3.9}$$

where g^3 and f^2 are the eigenvalues of (3.9) with respect to G^3 and F^2 ; and y is the eigenvalue of (3.9) with respect to the operator $Y(\equiv \frac{2}{\sqrt{3}} M$ of equation (3.3)).

The operators I_3 and Y correspond to the two mutually commuting operators denoted by H_1 and H_2 in the general theory, while $I_\pm$ $K_\pm$ and $L_\pm$ are losely related to $E_{\pm 1}$, $E_{\pm 2}$, $E_{\pm 3}$ of the general theory. To see this we write down the commutation relations, which are conveniently expressed with the aid of the unit vectors

$$\underline{i} = (1, 0) , \quad \underline{k} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) , \quad \underline{\ell} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

in the form,

$$
\begin{aligned}
[H_j, I_\pm] &= \pm i_j I_\pm \\
[H_j, K_\pm] &= \pm k_j K_\pm \\
[H_j, L_\pm] &= \pm \ell_j L_\pm \\
[I_+, I_-] &= 2 i_j H_j = 2 I_3 \\
[K_+, K_-] &= 2 k_j H_j = 2 K_3 \\
[L_+, L_-] &= 2 \ell_j H_j = 2 L_3 \\
[I_-, K_+] &= L_+ , \quad [K_-, I_+] = L_- \\
[L_-, I_-] &= K_- , \quad [I_+, L_+] = K_+ \\
[K_+, L_-] &= I_+ , \quad [L_+, K_-] = I_-
\end{aligned}
\tag{3.10}
$$

Here we have put $(I_3, Y) = (H_1, H_2)$. It is clear that $\underline{i}$, $\underline{k}$, $\underline{\ell}$ are proportional to the root vectors $\underline{\alpha}(1)$, $\underline{\alpha}(2)$, $\underline{\alpha}(3)$ of E_1, E_2, E_3.

For SU(3) we can denote an irreducible representation by $D(p, q)$ where $p = f_1 - f_2$ and $q = f_2 - f_3$.

The dimension formula may also be expressed in terms of p and q. Thus

$$
\text{Dim. } D(p, q) = \frac{1}{2}(p+1)(q+1)(p+q+2) \tag{3.11}
$$

Thus the matrices (3, 3) generate the 3-dimensional representation D(1, 0) while their negatives generate the 3^* representation D(0, 1). The 1-dimensional representation D(0, 0) has zero matrices for its generators. An arbitrary representation D(p, q) can be constructed from the direct product of p-representations D(1, 0) and q-representations D(0, 1).

The decomposition

$$D(p,q) \times D(p',q') = \sum_{p''q''} m_{p''q''} D(p'',q'') \tag{3.12}$$

is called the Clebsch-Gordan series; $m_{p''q''}$ being a positive integer. In the expression (3.9) we may replace f and g by p and q as these convey the same information.

Whenever $m_{p''q''}$ is non zero in (3.12) we will have a relation

$$\begin{aligned} &|p'',q'';i'',i_3'',Y''\rangle_\alpha \\ &= C_\alpha(p'',q'';i'',i_3'',Y''\,|\,p,q;i,i_3,Y\,;\,;p',q';i',i_3',Y') \\ &\quad \times |p,q;i,i_3,Y\rangle |p',q';i',i_3',Y'\rangle \end{aligned} \tag{3.12}$$

The suffix α is important when $m_{p''q''}$ is greater than 1; it represents a degeneracy in the way the representation D(p", q") is formed.

The coefficient C_α(-------) is called a Clebsch-Gordan coefficient for SU(3).

It turns out that the Clebsch-Gordan coefficients factorize. Thus

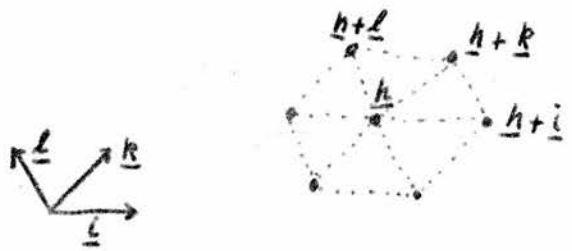
$$C_\alpha(p,q;i,i_3,y \mid p',q';i',i_3',y';;p'',q'';i'',i_3'',y'')$$
$$= (p,q;i,y \mid p',q';i',y';;p''q'';i'',y'')_\alpha$$
$$\times\, C(i\; i_3 \mid i',i_3';i'',i_3'') \tag{3.14}$$

where the two factors on the right hand side are called Iso-scalar factors[13] and Clebsch-Gordan Coefficients for SU(2) respectively; the former being so called because it does not depend on i_3, i_3' and i_3''.

A knowledge of the eigenstates (3.9) of a representation and the commutation relations (3.10) will enable us to construct the generators of the representation. Suppose we have a state with eigenvalues (h_1, h_2) with respect to (I_3, Y) (we denote them by a vector $\underline{h}$) then by consideration of the top 3 equations of (3.10) we have

$$\begin{aligned} \underline{H}\{I_\pm\, \phi(\underline{h})\} &= (\underline{h} \pm \underline{i})\{I_\pm\, \phi(\underline{h})\} \\ \underline{H}\{K_\pm\, \phi(\underline{h})\} &= (\underline{h} \pm \underline{k})\{K_\pm\, \phi(\underline{h})\} \\ \underline{H}\{L_\pm\, \phi(\underline{h})\} &= (\underline{h} \pm \underline{l})\{L_\pm\, \phi(\underline{h})\} \end{aligned} \tag{3.15}$$

where $\phi(\underline{h})$ denotes the state with eigenvalues $\underline{h}$. If none of $I_\pm\phi(\underline{h})$, $K_\pm\phi(\underline{h})$ $L_\pm\phi(\underline{h})$ are zero we have 6 new states. We can represent the situation graphically, in a weight diagram.



By repeated application of the operators $I_{\pm}$, $K_{\pm}$, $L_{\pm}$ we generate all the states. It is evident from the diagram above that when it is completed with all states, we will be unable to obtain a new state by a translation away from the outer edge. That is one of the operators $I_{\pm}$, $K_{\pm}$, $L_{\pm}$ will annihilate the state.

When all the commutation relations (3.10) are taken into account severe restrictions are imposed by the requirement that the number of states should be finite. To obtain these one introduces the notion of the highest weight. This is the vector $\underline{h}$ (or weight $\underline{h}$) belonging to the state such that, compared with any other weight $\underline{h}'$, then the first non-vanishing number in the series h_1, $- h_1'$, $h_2 - h_2'$ is positive. Clearly the operator I_+ annihilates this state. We find that it has the eigenvalue $h_1(h_1 + 1)$ of I^2 and that all the states

$$I_- \phi(\underline{h}) , I_- I_- \phi(\underline{h}), \ldots\ldots, \underbrace{I_- I_- \ldots I_-}_{2h_1} \phi(\underline{h})$$

exist but the next number of the sequence vanishes. This fixes h_1 to be integer of half-integer. The operators $\underline{K}_{\pm}$, $L_{\pm}$ do not commute with I^2.

The effect of these operators is summarised in the following table for states represented by points in the outer perimeter of the weight diagram and for $I_{\pm}$ in general.

Operator	Δi_3	ΔY	Δi
I_+	+1	0	0
I_-	-1	0	0
K_+	½	1	+½ at $i_3 = i$
K_-	-½	-1	+½ at $i_3 = -i$
L_+	-½	1	+½ at $i_3 = -i$
L_-	½	-1	+½ at $i_3 = i$

Note that there may be more than one state associated with a position on the weight diagram. It turns out that there is just one state to each position in the outer perimeter; two states to each position of the perimeter of the remaining states; three to each position in the next perimeter and so on.

The above table does not give sufficient freedom within a weight diagram to move from one position to any other. This can be remedied by the introduction of an operator

$$\bar{K}_- = 2K_- . I_3 + L_+ . I_- \tag{3.16}$$

Acting on a state with $i_3 = i$ with $\bar{K}_-$ sends i to $1 - ½$, i_3 to $i_3 - ½$ and Y to $Y - 1$. We have denoted this operator by $\bar{K}_-$ because it results in a state which is to be placed in the position shifted by $-\underline{k}$ from the state on which it acts.

The fourth equation of (3.10) forced us to choose the eigenvalues of I_3 integer or half-integer. The fifth and sixth equations demand that Y have integer eigenvalues.

We give below two tables showing how particles are associated with states of SU(3).

The Octet of Pseudoscalar Mesons

$\lvert p,q\,;\,i,i_3,Y\rangle$	Particle Operator	
$\lvert 1,1\,;\,\tfrac{1}{2},\tfrac{1}{2},1\rangle$	K^+	
$\lvert 1,1\,;\,\tfrac{1}{2},-\tfrac{1}{2},1\rangle$	K^o	
$\lvert 1,1\,;\,1,1,0\rangle$	$-\pi^+$	
$\lvert 1,1\,;\,1,0,0\rangle$	π^o	
$\lvert 1,1\,;\,1,-1,0\rangle$	π^-	
$\lvert 1,1\,;\,0,0,0\rangle$	η	
$\lvert 1,1\,;\,\tfrac{1}{2},\tfrac{1}{2},-1\rangle$	$-\overline{K}^o$	
$\lvert 1,1\,;\,\tfrac{1}{2},-\tfrac{1}{2},-1\rangle$	K^-	

Spin $\frac{3}{2}$ Baryon Decouplet

$\|p,q;\ i,i_3,y\rangle$	Particle Operator	
$\|3,0;\ \frac{3}{2},\ \frac{3}{2},\ 1\rangle$	N^{*++}	
$\|3,0;\ \frac{3}{2},\ \frac{1}{2},\ 1\rangle$	N^{*+}	
$\|3,0;\ \frac{3}{2},-\frac{1}{2},\ 1\rangle$	N^{*o}	
$\|3,0;\ \frac{3}{2},-\frac{3}{2},\ 1\rangle$	N^{*-}	
$\|3,0;\ 1,\ 1,\ 0\rangle$	X^{*+}	
$\|3,0;\ 1,\ 0,\ 0\rangle$	X^{*o}	
$\|3,0;\ 1,-1,\ 0\rangle$	X^{*-}	
$\|3,0;\ \frac{1}{2},\ \frac{1}{2},\ -1\rangle$	Ξ^{*o}	
$\|3,0;\ \frac{1}{2},\ -\frac{1}{2},\ -1\rangle$	Ξ^{*-}	
$\|3,0;\ 0,\ 0,\ -2\rangle$	Ω^-	

The symbols +, o, - refer to the charge of the particles, while * is to distinguish them from other particles similarly denoted.

Instead of working with representations of various dimensions we can as in the previous chapter work with irreducible tensors. When this is done, we have to know the correspondence between the components of such tensors, and the physical states or particle operators. The Decouplet above is represented by a completely symmetric rank 3 tensor, D_{ijk}. The following correspondence can be made with particle operators.

$$D_{333} = \Omega^- \ , \ D_{133} = \tfrac{1}{\sqrt{3}}\,\Xi^{*0} \ , \ D_{233} = \tfrac{1}{\sqrt{3}}\,\Xi^{*-}$$

$$D_{113} = \tfrac{1}{\sqrt{3}}\,X^{*+} \ , \ D_{123} = \tfrac{1}{\sqrt{6}}\,X^{*0} \ , \ D_{223} = \tfrac{1}{\sqrt{3}}\,X^{*-}$$

$$D_{111} = N^{*++} \ , \ D_{112} = \tfrac{1}{\sqrt{3}}\,N^{*+} \ , \ D_{122} = \tfrac{1}{\sqrt{3}}\,N^{*0}$$

$$D_{222} = N^{*-} \tag{3.17}$$

Other states D_{ijk} are obtained from the symmetry relations

$$D_{ijk} = D_{jki} = D_{jik} \qquad \text{etc.}$$

The normalisation factors, 1, $1/\sqrt{3}$, $1/\sqrt{6}$ ensure ~~d~~ the result

$$\sum_{ijk} D^{+ijk} D_{ijk} = \sum_X X^+ X$$

where X runs over the 10 states Ω^-, Ξ^{*0}, ..., N^{*-}

In order to determine eigenvalues of a particular state we have to know what the generators F_i of SU(3) become for higher representations. Since, however, we are working in terms of tensors we only have to consider what the Cartesian operator for a tensor of rank f becomes. By considering infinitesimal group rotations $1 + \alpha_i F_i$, we have

$$\underbrace{(1 + \alpha_i F_i) \times (1 + \alpha_i F_i) \times \ldots\ldots \times (1 + \alpha_i F_i)}_{(\text{f terms})} = 1 + \alpha_i F_i^{(f)}$$

or

$$F_i^{(f)} = F_i \times 1 \times \ldots \times 1 + 1 \times F_i \times 1 \times \ldots \times 1 + \ldots$$
$$\ldots + 1 \times 1 \times \ldots \times 1 \times F_i .$$

where $F_i^{(f)}$ is the generator of the Cartesian rank f representation corresponding to F_i.

Suppose we wanted to know if D_{111} is an eigenstate of I_3; we proceed

$$(I_3)_1^{\alpha} D_{\alpha 11} + (I_3)_1^{\alpha} D_{1\alpha 1} + (I_3)_1^{\alpha} D_{11\alpha}$$

$$= 3(I_3)_1^{\alpha} D_{\alpha 11} \text{ (by symmetry of } D_{\alpha\beta\gamma})$$

$$= \frac{3}{2} D_{111} \text{ (from equation (3.3)).}$$

Thus, D_{111} is an eigenstate of I_3 with eigenvalue $i_3 = {}^3/2$.

In addition to the meson Octet there is also of importance, the baryon octet of spin ½ particles, and the octet of spin 1 mesons. These sets of particles are distinguished from other sets by their baryon-number and their spin. This suggests enlarging the group with which we are dealing to

$$SU(2) \times U(1) \times SU(3) \tag{3.18}$$

so that every particle state is now specified by its spin quantum numbers (SU(2)), its baryon number (U(1)) and its internal symmetry quantum numbers. In the next chapter we see how the group (3.18) is a subgroup of SU(6) and see what predictions result from the assumption that interactions are SU(6) invariant, which were not already a consequence of invariance under SU(2) × U(3).

The Breaking of SU(3).

The particles we have classified into representations of SU(3) do not have identical properties; for instance the masses within a given multiplet may vary by amounts comparable with the mean mass of the multiplet. The isospin submultiplets, however, have mass variations of only a few million electron volts.

In order to describe this situation we divide the strong interactions into two kinds:

1. Very strong interactions which are invariant under SU(3), and

2. Medium strong interactions which break SU(3) but which leave the isospin subgroup invariant.

We would like to introduce an operator M, which we will call the mass operator such that when operated on a given state will yield its mass. If this operator can be written as a linear combination of the generators of SU(3) we may hope that, in view of the invariance of the SU(2) subgroup we will have fewer free parameters than particles in a given representation. To show this we introduce the concept of a Tensor Operator. An irreducible tensor Operator $T(p,q)$ is a set of operators which transform like the $\frac{1}{2}(p+1)(q+1)(p+q+2)$ components of a tensor under $D(p,q)$. If we denote the components of such a tensor by t^{μ}_{ν} where μ stands in place of (p,q), then we have

$$F_i \, t^{\mu}_{\nu} = C_{i\nu\nu'} \, t^{\mu}_{\nu'} \tag{3.19}$$

We may define our tensor operator as a set of operators T^{μ}_{ν} such that

$$[F_i \, , T^{\mu}_{\nu}] = C_{i\nu\nu'} \, T^{\mu}_{\nu'} \tag{3.20}$$

If we had been discussing SU(2) we would have tensor operators T^{j}_{m}. For SU(2) there is a Theorem, the Wigner-Eckart Theorem which states that the matrix element of a tensor operator

T^k_q can be reduced to a product of a Clebsch-Gordan coefficient and a reduced matrix element which is independent of the 'magnetic' quantum numbers, m.

$$\langle j', m' | T^k_q | j, m \rangle$$

$$= C(j, m | k, q ; j', m') \times \frac{\langle j' \| T(k) \| j \rangle}{(2k+1)^{\frac{1}{2}}} \tag{3.21}$$

For SU(3) there is a similar result

$$\left\langle \begin{matrix} \mu' \\ \nu' \end{matrix} \middle| T^{\mu}_{\nu} \middle| \begin{matrix} \mu'' \\ \nu'' \end{matrix} \right\rangle$$

$$= \sum_{\alpha} \begin{pmatrix} \mu' & \mu & \mu'' \\ \nu' & \nu & \nu'' \end{pmatrix}_{\alpha} \langle \mu' \| T(\mu) \| \mu'' \rangle \tag{3.22}$$

the summation over α arising from the degeneracy in the way a tensor of particular symmetry type is formed.

If the tensor T^{μ}_{ν} is a scalar operator S, (3.22) becomes

$$\left\langle \begin{matrix} \mu' \\ \nu' \end{matrix} \middle| S \middle| \begin{matrix} \mu'' \\ \nu'' \end{matrix} \right\rangle = \delta_{\mu'\mu''} \delta_{\nu'\nu''} f(\mu') \tag{3.23}$$

As a second example we consider the 'octet' operator. This is a set of 8 tensors transforming as an 8 component tensor of SU(3); F_i of equation (3.3) form such an operator.

We could take as definition of an octet operator, the equation

$$[A_i^{\,j}, T_l^{\,m}] = \delta_i^{\,m} T_l^{\,j} - \delta_l^{\,j} T_i^{\,m} \tag{3.24}$$

It is evident that $A_i^{\,j}$ are themselves an octet operator.

We can form a second octet operator from two $A_i^{\,j}$ (or F's) by compounding them with the Clebsch-Gordan coefficients. In matrix notation, the resulting octet operator D_i^j is given by

$$D_i^j = \sqrt{\tfrac{3}{2}}\left(A_k^j A_i^k + A_i^k A_k^j\right) - \sqrt{\tfrac{2}{3}}\,\delta_i^j\left(A_k^l A_l^k\right) \tag{3.25}$$

Any other octet operator is a linear combination of F_i^j and D_i^j. We note

$$\begin{aligned} F^2 &= \tfrac{1}{2} A_i^j A_j^i \\ G^3 &= \tfrac{1}{\sqrt{6}} D_i^j A_j^i \end{aligned} \tag{3.26}$$

After making use of the freedom associated with α in the definition of the Clebsch-Gordan coefficients, we have

$$\begin{aligned} \left\langle {N' \atop \nu'} \right| T_\nu^{(8)} \left| {N'' \atop \nu''} \right\rangle &= \sum_{\alpha=1}^{2} \begin{pmatrix} N' & (8) & N'' \\ \nu' & \nu & \nu'' \end{pmatrix}_\alpha \langle N|T|N\rangle_\alpha \\ &= A(N)\left\langle {N' \atop \nu'} \right| F_\nu \left| {N' \atop \nu''} \right\rangle + B(N)\left\langle {N' \atop \nu'} \right| D_\nu \left| {N' \atop \nu''} \right\rangle \end{aligned} \tag{3.27}$$

where F_ν and D_ν denote the component of the tensor operators F_i (or A_i^j) and D_i^j associated with the quantum numbers ν.

If $\nu = (0, 0, 0)$ then we must take $\nu'' = \nu'$. Now

$$F_{(0,0,0)} = \sqrt{\tfrac{3}{2}}\, Y$$

and

$$D_{(0,0,0)} = 3I^2 - \tfrac{3}{4}Y^2 - F^2$$

so that

$$\left\langle {N \atop \nu} \right| T_{(0,0,0)}^{(8)} \left| {N \atop \nu} \right\rangle = a_1 y + a_2\left[i(i+1) - \tfrac{1}{4}y^2 - \tfrac{1}{3}f^2\right]$$

More generally

$$\left\langle {N \atop \nu} \right| T_{(0,0,0)}^{(1)} + T_{(0,0,0)}^{(8)} \left| {N \atop \nu} \right\rangle$$

$$= a + b\,y + c\left[i(i+1) - \tfrac{1}{4}y^2\right] \tag{3.28}$$

Equation (3.28) is the basis of Gell-Mann - Okubo mass formula[4].

We require an operator which commutes with I^2, I and Y. This can be compounded by forming a linear combination of the components of tensor operators with $\nu = (0, 0, 0)$. Such tensors have dimensions 1, 8, 27, 64,... . We may thus expect to express the mass operator as[12]

$$M = T^{(1)}_{(0,0,0)} + T^{(8)}_{(0,0,0)} + T^{(27)}_{(0,0,0)} + \ldots \tag{3.29}$$

The first term $T^{(1)}_{(0,0,0)}$ is invariant for SU(3) interactions, while $T^{(8)}_{(0,0,0)}$ and higher terms break SU(3). If we suppose that only the first two terms of (3.29) are of importance we obtain

$$\langle {}^{\mu}_{\nu} | M | {}^{\mu}_{\nu} \rangle = M_0 + M_1 y + M_2 [i(i+1) - \tfrac{1}{4} y^2] \tag{3.30}$$

If we apply this formula to the baryon octet we are left, after elimination of M_0, M_1, M_2 from the resulting equations, with the relation

$$\frac{1}{2}(M_N + M_{\Xi}) = \frac{3}{4} M_{\Lambda} + \frac{1}{4} M_{\Sigma} \tag{3.31}$$

which connects the masses of the isospin sub-multiplets.

For Boson octets we interpret M as a mass squared operator rather than a mass operator; this is justified by the way the different fields occur in the Lagrangians. In consequence we have an operator with

$$\langle {}^{\mu}_{\nu} | M^2 | {}^{\mu}_{\nu} \rangle = m_0^2 + m_1^2 [i(i+1) - \tfrac{1}{4} y^2] \tag{3.32}$$

There is no term linear in y on account of the fact that the square of the mass of the antiparticle will be the same as for the particle.

By similar analysis it is also possible to obtain relations for the electromagnetic mass splitting and to obtain relations between the values of the magnetic moments of the particles. To do this we note that the decomposition of SU(3) with respect to the isospin group is not suited to the problem; instead decomposition relative to the L-subgroup generated by L_1, L_2, L_3 is appropriate. More predictions of this type can be obtained from the group SU(6) which we discuss in the next chapter.

CHAPTER 4.

SU(6) SYMMETRY.

So far we have considered a classification of particles with the aid of SU(3). In this way we have been lead to regard, what was previously thought of as a set of particles, as different states of the same particle. The different states are distinguished by the internal quantum numbers of SU(3) and the baryon number B. It is also necessary to specify the spin of the states.

When this is done we have not specified the particle states completely as we still have undecided the direction of spin of the particle and its momentum. This suggests that particle states should be classified under $SU(2) \times U(3)$ or some larger group with $SU(2) \times U(3)$ as a subgroup; we leave aside for the moment the question of translational invariance (i.e. momentum).

If we consider an interaction Lagrangian written with the aid of the generators of SU(3), which we will denote now by T^i, we find occurring together between field operators, products of the type $\gamma_\mu T^i$, $i\gamma_5\gamma_\mu T^i$, $\sigma^{\mu\nu} T^i$ etc. The indices will be summed with respect to their occurrence elsewhere in the equation. In the non-relativistic situation we would expect products of the type T^i, $\sigma^\ell T^i$, etc. These products are in effect direct products of the generators of SU(2) with those of SU(3). The resulting interaction will be invariant under $SU(2) \times SU(3)$. It was postulated by F. Gürsey and L.A. Radicati(1) that the group of invariance should be extended to the group with generators $\sigma^\ell \times T^i$, $\sigma^\ell \times 1$, $1 \times T^i$. This is the group SU(6). The generators $\sigma^\ell \times 1$, $1 \times T^i$ are necessary in addition to $\sigma^\ell \times T^i$ as these by themselves do not generate a group.

When considering SU(3) the 3 dimensional representation D(1,0) can be filled with 3 spin ½ particles called quarks. These particles have not been observed experimentally. It is supposed that they are very massive, they have fractional charge and baryon number $\frac{1}{3}$. Whether or not these particles exist is not crucial to the theory but it is convenient to regard other particles as bound states of this basis set and their antiparticles. Particles transforming under the representation D(p,q) of SU(3) may be thought of as bound states of p quarks and q antiquarks; the baryon number is then $\frac{1}{3}(p-q)$, but this is not essential as $B = (f_1+f_2+f_3) = \frac{1}{3}(p-q)+f_2$.

For SU(6) the basic representation is again filled by the quarks, but when spin is taken into account there are of course 6 different states. In order to decide which states occupy a given representation of SU(6) we would like first to know its decomposition under the subgroup SU(2) × SU(3). There is ambiguity in the way in which a subgroup SU(2) × SU(3) may be chosen; we must choose this subgroup such that the basic six component representation of SU(6) contains this subgroup as the matrix direct product of the basic 2-dimensional representation of SU(2) and the basic 3-dimensional representation of SU(3). We write this

$$6 = (2, 3) \tag{4.1}$$

If we put $\lambda_1 = f_1-f_2$, $\lambda_2 = f_2-f_3, \ldots\ldots, \lambda_5 = f_5-f_6$. The representation of SU(6) considered may be denoted $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$. The 6 dimensional representation above is in this notation (1, 0, 0, 0, 0) while its adjoint 6 is (0, 0, 0, 0, 1). In general the adjoint representation is

$(\lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1)$. If we can find the 'content' of all the single column tableaux (symmetry patterns) of SU(6) we will easily be able to find the content of any other by considering the decomposition of that tableau in terms of the column tableaux. For SU(6) the following procedure works.

From

$$\square \times \square = \begin{smallmatrix}\square\\\square\end{smallmatrix} + \square\square$$

$$\text{or} \quad 6 \times 6 = 15 + 21$$

we see that the contents of the 6×6 must divide into 15 and 21 states. The contents of 6×6 is

$$\begin{aligned} & (2, 3) \times (2,3) \\ = & \big((3 + 1), (6 + \overset{*}{3})\big) \\ = & (3, 6) + (3, \overset{*}{3}) + (1, 6) + (1, \overset{*}{3}) \quad . \end{aligned}$$

There is only one way in which this can be done, namely:

$$\begin{aligned} 15 &= (3, \overset{*}{3}) + (1, 6) \\ 21 &= (3, 6) + (1, \overset{*}{3}) \end{aligned} \tag{4.2}$$

Now consider

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \square = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} + \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$$

$$\text{or} \quad 15 \times 6 = 20 + 76 \quad .$$

The contents of 15×6 is :

$$\begin{aligned} & ((3, 3^*) + (1, 6)) \times (2, 3) \\ = & (4 + 2, 1 + 8) + (2, 10 + 8) \\ = & (4, 1) + (4, 8) + (2, 1) + (2, 8) + (2, 8) + (2, 10) \ . \end{aligned}$$

From the fact that the antisymmetric 20 of SU(6) is self-adjoint we must have

$$20 \;=\; (2,\ 8) + (4,\ 1) \qquad (4.3)$$

and hence

$$70 \;=\; (4,\ 8) + (2,\ 1) + (2,\ 8) + (2,\ 10)$$

We have used the fact that in SU(3) $8^* = 8$, $1^* = 1$, but $10^* \neq 10$.

The contents of the other 3 column representations of SU(3) are obtained by taking the adjoints of (4, 1), (4, 2) and (4, 3). Thus

$$\begin{aligned} 6 &= (2,\ 3) \\ 15 &= (3,\ 3^*) + (1,\ 6) \\ 20 &= (2,\ 8) + (4,\ 1) \end{aligned} \qquad (4.4)$$

Some representations of physical interest are:

$(\lambda_1, \ldots, \lambda_5)$	Dimension	Contents (SU(2), SU(3))
(1, 0, 0, 0, 1)	35	(3, 8), (1, 8), (3, 1),
(3, 0, 0, 0, 0)	56	(4, 10), (2, 8)
(1, 1, 0, 0, 0)	70	(2, 10), (4, 8), (2, 8),(2, 1

The 35 dimensional representation arises from the decomposition

$$6 \times 6 \;=\; 35 + 1 \;.$$

We may expect it therefore to contain meson states. We see from its decomposition that it contains an octet of spin 0 and an octet of spin 1 mesons and also a spin 1 singlet.

The 56, the 70, and also the 20 arise in the decomposition

$$6 \times 6 \times 6 \quad = \quad 20 + 70 + 70 + 56$$

and so all could represent particles of baryon number 1.

The 10 spin $^3/2$ baryons and the 8 spin ½ baryons are assigned to the 56[1].

In tensor notation the baryons are represented by a symmetric tensor B_{ABC} with antibaryons represented by $\overline{B}^{ABC}$ and mesons by a traceless tensor $M_A{}^B$, where the indices run from 1 to 6.

The mesons can couple to the baryon-antibaryon structure in just one way. In order to couple the mesons to the baryons we need to be able to construct a 35 component representation from the baryons and antibaryons. This will be possible only if a 35 appears in the decomposition 56 × 56 . This is the case

$$56 \times 56 \quad = \quad 1 + 35 + 405 + 2695 \tag{4.5}$$

There is only one way of doing this as 35 occurs in the decomposition just once.

The coupling may be represented

$$g\, \overline{B}^{ABC}\, B_{A'BC}\, M_A{}^{A'} \tag{4.6}$$

The actual 35 component object coupling to $M_A{}^B$ is

$$J_A{}^{A'} \quad = \quad \overline{B}^{A'BC}\, B_{ABC} \;-\; \delta_A^{A'} \langle \overline{B}\, B \rangle \tag{4.7}$$

When considering the SU(2) × SU(3) subgroup the indices A, B, C separate into pairs $A \equiv (i,\alpha)$; $B \equiv (j,\beta)$; $C \equiv (k,\gamma)$, where i, j, k are SU(2) indices and take the values 1 and 2 and α, β, γ are SU(3) indices and take the values 1, 2 and 3. Under this subgroup B_{ABC} decomposes

$$B_{ABC} \equiv B_{i\alpha, j\beta, k\gamma}$$

$$= D_{\alpha\beta\gamma, ijk}$$

$$+ \frac{1}{3\sqrt{2}} \Big[\epsilon_{\alpha\beta\delta}\, \epsilon_{ij}\, N_{\gamma,k}^{\delta} + \epsilon_{\beta\gamma\delta}\, \epsilon_{jk}\, N_{\alpha,i}^{\delta}$$

$$+ \epsilon_{\gamma\alpha\delta}\, \epsilon_{ki}\, N_{\beta,j}^{\delta} \Big] \qquad (4.8)$$

where $D_{\alpha\beta\gamma, ijk}$ is symmetric in both Latin and Greek indices and is the tensor corresponding to (4,10) in the decomposition

$$56 = (2, 8) + (4, 10) \quad .$$

$N^{\beta}_{\alpha, i}$ is an octet of SU(3) and a doublet of SU(2). The normalisation is explained in Appendix B.

Substituting (4.8) into (4.7) we obtain

$$J_A^{\ A'} \equiv J_{i\alpha}^{\ i'\alpha'} = \bar{D}^{\alpha'\beta\gamma, i'jk} D_{\alpha\beta\gamma, ijk} - \frac{1}{6} \delta_\alpha^{\alpha'} \delta_i^{i'} \langle \bar{D} D \rangle$$

$$+ \frac{\sqrt{2}}{3} \Big[\epsilon^{\alpha'\beta\delta} \epsilon^{i'j} \bar{N}_\delta^{\gamma, k} D_{\alpha\beta\gamma, ijk}$$

$$+ \epsilon_{\alpha\beta\delta}\, \epsilon_{ij}\, \bar{D}^{\alpha'\beta\gamma, i'jk} N_{\gamma,k}^{\delta} \Big]$$

$$+ \frac{1}{6} \Big\{ \delta_i^{i'} (\bar{N} N_F)_\alpha^{\alpha'} + \frac{2}{3} (\sigma^\nu)_i^{i'} \big[3 (\bar{N} \sigma^\nu N_D)_\alpha^{\alpha'}$$

$$+ 2 (\bar{N} \sigma^\nu N_F)_\alpha^{\alpha'} + \delta_\alpha^{\alpha'} \langle \bar{N} \sigma^\nu N \rangle \big] \Big\} \qquad (4.9)$$

where σ^ν are the Pauli matrices,

$$(\bar{N} N_F)_\alpha^{\alpha'} = \bar{N}_\beta^{\alpha'} N_\alpha^\beta - \bar{N}_\alpha^\beta N_\beta^{\alpha'}$$

$$(\bar{N} N_D)_\alpha^{\alpha'} = \bar{N}_\beta^{\alpha'} N_\alpha^\beta + \bar{N}_\alpha^\beta N_\beta^{\alpha'} \qquad (4.10)$$

The F and D of equations (4.9), (4.10) signify a connection with the F and D type octet tensor operators. Since the 35 appears only once in the decomposition $56 \times 56^{*}$ the F/D ratio is fixed; that is the proportion of F-type to D-type coupling in equation (4.9) is determined.

As we will see this will enable us to predict the ratio of neutron to proton magnetic moment. This was not possible for SU(3) where the corresponding decomposition is

$$8 \times 8 = 27 + 10 + \overline{10} + 8 + 8 + 1 \quad .$$

The appearance of the 8 twice in this decomposition leaves the F/D ratio indeterminate.

The magnetic moment of nucleon arises out of the interaction with the electromagnetic field. We assume that the electromagnetic field transforms like a component of the 35 (regular) representation of SU(6). This corresponds to procedure of expanding the mass breaking terms of SU(3) in terms of operators of the regular octet and higher order tensors.

For SU(3) the charge operator is

$$Q_{\alpha}^{\alpha'} = e\left(I_3 + Y\right) \tag{4.11}$$

Contracting $Q_A^{A'} = \delta_i^{i'} Q_\alpha^{\alpha'}$ with (4.9) will yield the charge and $M_A^{A'} = Q_\alpha^{\alpha'} (\vec{\sigma})_i^{i'}$ with (4.9) the magnetic moment ratios of the particles involved.

The operator Q was of course chosen to give the correct charge so only the magnetic moment ratio is of interest. The decomposition of the 35

$$35 = (3, 8) + (1, 8) + (3, 1)$$

represents, in this context, a magnetic current term (3.8) and an electric current term (1, 8). The magnetic moments of the particles will be proportional to the couplings of $M_A^{A'}$ to the (3, 8) term of (4, 9).

With the aid of the explicit form(14)

$$N_\alpha{}^\beta = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & -p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & -n \\ \Xi^- & -\Xi^0 & -\frac{2\Lambda^0}{\sqrt{6}} \end{pmatrix} \tag{4.12}$$

we readily obtain

$$\mu_p / \mu_n = -3/2 \tag{4.13}$$

It is also possible to obtain the electromagnetic mass splittings by these methods. Because this splitting is second order we will expect the splitting to depend on

$$35 \times 35 = 1 + 35 + 35 + 189 + 280 + 280 + 405 \tag{4.14}$$

This must couple to the product $\overline{B}\,B$ of the baryons, i.e. to terms of the type

$$56^{*} \times 56 = 1 + 35 + 405 + 2695 \qquad (4.15)$$

In (4.14) only one of the 35 is **relevant** as one is symmetric and one antisymmetric in the 35-tensors from which they were formed. Only the symmetric one must couple. The scalar tensor 1 of course does not lead to mass splitting. We are thus left with two ways of coupling (4.14) and (4.15), namely through the 35 and the 405. Since the operator $Q_\alpha^\beta = e\left(\delta_{1\alpha}\delta_{1\beta} - \frac{1}{3}\delta_{\alpha\beta}\right)$, the electromagnetic mass splitting is of the form

$$\alpha\, \bar{B}^{(i,1),A,\beta} B_{(i,1),A,\beta} + \beta\, \bar{B}^{(i,1),(j,1),A} B_{(i,1),(j,1),A} \qquad (4.16)$$

The first term here involves a contribution from both 35 and 405 while the second is purely from 405.

Substituting (4.8) into (4.16) ~~thus~~ we obtain

$$\alpha \left\{ \bar{D}^{1\alpha\beta} D_{1\alpha\beta} + \frac{1}{3}\left(\bar{N} N_F\right)_1^1 + \frac{1}{3}\langle \bar{N} N \rangle \right\}$$

$$+ \beta \left\{ \bar{D}^{11\alpha} D_{11\alpha} + \frac{1}{3}\left(\bar{N}_\alpha^1 N_1^\alpha - \bar{N}_1^1 N_1^1 \right) \right\} \qquad (4.17)$$

Working this out explicitly and eliminating α and β leads to:

$$N^{*+} - N^{*0} = Y^{*+} - Y^{*0} = p - n = \Sigma^{+} - \Sigma^{0}$$

$$\Xi^{*-} - \Xi^{*0} = N^{*-} - N^{*0} = Y^{*-} - Y^{*0}$$

$$= \Xi^{-} - \Xi^{0} = \Sigma^{-} - \Sigma^{0}$$

$$N^{*++} = 3\left(N^{*+} - N^{*0}\right) + N^{*-}$$

(4.18)

where we have denoted the particle mass by its symbol. Notice that the results are expressed in terms of differences of particle masses within the same isomultiplet. We suppose that the effect medium strong interactions is not important. It is also possible to obtain a mass-formula for SU(6) as was done for SU(3) in the last chapter, the essential change in the result is an extra term $M_3\ j(j+1)$ in (3 30) or $m_3^2\ j(j+1)$ in (3 32).

In obtaining the result (4.13) we have neglected the fact that the actual values μ_p and μ_n are obtainable from the analysis. These results are, however, not in good agreement with experiment while (4.13) is in excellent agreement.

In the next chapter we show how to rectify this situation by considering the relativistic analogy. We also show that there is no need to include the entire internal group SU(3) to obtain results for non-strange particles.

CHAPTER 5

THE COVARIANT COMBINATION OF SPIN INDEPENDENCE AND INTERNAL SYMMETRY.

In forming a relativistic group we might try forming the direct product of the internal generators, T^i and the generators of the Lorentz group $\sigma^{\mu\nu}$, and use the matrices $T^i \times \sigma^{\mu\nu}$ to generate a group.

Unfortunately these matrices do not form a group except under the unrealistic assumption that the T^i commute. This situation can be understood from the equation

$$[T^i\sigma^{\mu\nu}, T^j\sigma^{\lambda\rho}]$$

$$= \frac{1}{2}[T^i, T^j]\{\sigma^{\mu\nu}, \sigma^{\lambda\rho}\} + \frac{1}{2}\{T^i, T^j\}[\sigma^{\mu\nu}, \sigma^{\lambda\rho}] \quad (5.1)$$

It is evident from this equation that the direct products of generators of two non-abelian groups are generators of a group if and only if the anti-commutators of the generators belong to the algebra of the generators. We can always choose a pair of generators which do not commute but do anti-commute.

With this in mind we easily see how to amend the situation. We simply extend the groups by adding anticommutators and commutators until such time as we have a set closed under both commutation and anticommutation.

For a special unitary group SU(n) this means simply going to U(n), while a glance at the commutation and anticommutation relations (C. 6), (C. 1), (C. 2), (C.9) and (C.10) of Appendix C shows that we need add only the generators, I and γ_5 to $\sigma^{\mu\nu}$. The Lorentz group thus extended is isomorphic to the group $U(2)_L \times U_R(2)$.

If we combine this group with U(3) internal symmetry we obtain the group $W(6) \equiv U_L(6) \times U_R(6)$. This group has been discussed by Delbourgo, Salam and Strathdee[15] and by Feynman, Gell-Mann and Zweig[16]. It is shown in Appendix A that $\bar{\psi}\psi$ formed from Dirac spinors is not only invariant under the Lorentz group but also under the group $\tilde{U}(4)$ defined by equations (A.18) and (A.20). As we are unable to form a group from the matrices $\sigma^{\mu\nu} \times T^i$ we might guess that the full invariance of $\tilde{U}(4)$ should be combined with internal spin. That is, we consider the matrices $\Gamma^R \times T^i$ where $\Gamma^R \times T^i$ is defined by (A.18), and T^i are the generators of some suitably extended internal group. A theory whose starting point is invariance under the group $\tilde{U}(12)$, the group generated by $\Gamma^R \times T^i$ when T^i are the generators of U(3), has been developed by Delbourgo, Salam and Strathdee[3]. We discuss in detail the analogous theory $\tilde{U}(8)$ based on the internal symmetry SU(2).

$\tilde{U}(8)$ is clearly a subgroup of $\tilde{U}(12)$ and describes only non-strange particles. The point of considering this group is to distinguish between results arising from the assumption of spin independence and an almost perfectly observed internal symmetry and results which are complicated by the internal symmetry breaking of the strange particles. The manner in which $\tilde{U}(8)$ is broken is not confused by internal breaking. We shall show that the proton and neutron magnetic moments are already correctly predicted by $\tilde{U}(8)$ and depend in no way on its extension to $\tilde{U}(12)$. We anticipate that all the results obtainable from $\tilde{U}(12)$ concerning only non-strange particles are contained in the $\tilde{U}(8)$ theory.

Representations of $\tilde{U}(8)$.

The spinor representation of $\tilde{U}(8)$ is generated by the matrices

$$(\Gamma^R \times T^i)_A^B = (\Gamma^R)_\alpha^\beta (T^i)_p^q$$

where

$$A \equiv (\alpha, p) \quad , \quad B \equiv (\beta, q).$$

T^i are the 2 × 2 isotopic spin matrices supplemented by $T^o = I = 3B$, where B is the baryon operator for the spinor, or quark field. The matrices Γ^R are the 4 × 4 matrices,

$$I\,, \; \gamma^\mu\,, \; \sigma^{\mu\nu}\left(= \tfrac{i}{2}[\gamma^\mu, \gamma^\nu]\right)\,, \; i\gamma^\mu\gamma^5\,, \; \gamma^5$$

and have been discussed in Appendix A.

To complete the definition of the group we note that the quark field $\psi_A \equiv \psi_{p,\alpha}$ transforms as

$$\delta\psi_{p,\alpha} = i\epsilon^{iR} (\Gamma^R)_\alpha^\beta (T^i)_p^q \psi_{q,\beta} \tag{5.2}$$

with ϵ^{iR} assumed real.

With $\bar{\psi}$ defined by

$$\bar{\psi}^{p,\alpha} = \psi^+_{q,\beta} A^{\beta\alpha} \delta^{qp} \tag{5.3}$$

in analogy with $\bar{\psi}$ of Appendix A . We deduce that

$\bar{\psi}\,\psi$ is an invariant under $\tilde{U}(8)$.

Since also the total antisymmetric tensor in eight dimensions is a representation of $\tilde{U}(8)$, it follows that under $\tilde{SU}(8)$ $\bar{\psi}$ behaves like an antisymmetric combination of 7 quarks.

Young's tableaux techniques will be appropriate to $\tilde{U}(8)$ and, from the above, the equivalence of $\bar{\psi}$ to one of the tableaux shows that everything is entirely the same as for unitary groups.

We note the decompositions

$$\begin{aligned} 8 \times 8^* &= 1 + 63 \\ 8 \times 8 \times 8 &= 120 + 168 + 168 + 56 \quad . \end{aligned} \tag{5.4}$$

Here the 120 is completely symmetric, the 56 completely anti-symmetric and the 168 of symmetry type (2, 1).

Under the subgroup $U(2) \times \tilde{U}(4)$, the representations of $\tilde{U}(8)$ are reducible, having the following decompositions

$$\begin{aligned} 8 &= (2, 4) \\ 63 &= (3, 15) + (3, 1) + (1, 15) \\ 120 &= (4, 20) + (2, 20') \\ 168 &= (4, 20') + (2, 20) + (2, 20') + (2, 4^*) \\ 56 &= (2, 20') + (4, 4^*) \quad ; \end{aligned} \tag{5.5}$$

the pairs of bracketed numbers referring to the transformation properties under $U(2)$ and $\tilde{U}(4)$ respectively.

Here 20 is completely symmetric and 20' is of symmetry type (2, 1).

We assign the mesons to the self-adjoint 63 representation and the baryons to the symmetric 120. We shall thus be interested in the following decompositions

$$\begin{aligned} 63 \times 63 &= 1 + 63 + 63 + 120 + 945 + 945^* + 1{,}232 \\ 120 \times 120^* &= 1 + 63 + 1{,}232 + 13{,}104 \\ 63 \times 120 &= 120 + 168 + 2{,}520 + 4{,}752 \quad . \end{aligned} \tag{5.6}$$

Since in the decomposition of $120 \times 120^*$ the 63 is contained only once, there is just one coupling possible for the baryon-meson vertex. This is

$$\left[\bar{\psi}^{ABC}\psi_{A'BC} - \tfrac{1}{8}\langle\bar{\psi}\psi\rangle\delta_{A'}^{A}\right]\Phi_{A}^{A'}$$

where ψ_{ABC} represents the baryons and Φ_A^B the mesons.

The meson field may be expanded in terms of the matrices $(\Gamma^R T^i)_A^B$.

It will thus prove convenient to calculate

$$J^{Ri} = \left[\bar{\psi}^{ABC}\psi_{A'BC} - \tfrac{1}{8}\langle\bar{\psi}\psi\rangle\delta_{A'}^{A}\right](\Gamma^R T^i)_A^{A'} \tag{5.7}$$

Equation (5.7) can be rewritten

$$J^{Ri} = \bar{\psi}^{ABC}(\Gamma^R T^i)_A^{A'}\psi_{A'BC} \tag{5.8}$$

except when

$$(\Gamma^R T^i)_A^{A'} = \delta_A^{A'}$$

in which case J^{Ri} vanishes.

The completely symmetric tensor $\psi_{ABC} \equiv \psi_{\alpha p,\beta q,\gamma r}$ may be expanded as given by (5, 5),

$$\begin{aligned}\psi_{\alpha p,\beta q,\gamma r} &= D_{\alpha\beta\gamma,pqr} \\ &+ \left\{ \epsilon_{pq} N_{[\alpha\beta]\gamma,r} + \epsilon_{qr} N_{[\beta\gamma]\alpha,p} + \epsilon_{rp} N_{[\gamma\alpha]\beta,q} \right\}\end{aligned} \tag{5.9}$$

where $D_{\alpha\beta\gamma,pqr}$ is separately symmetric in $\alpha\beta\gamma$ and pqr.

$N_{[\alpha\beta]\gamma}$ is of symmetry type (2, 1) satisfying

$$\begin{aligned} N_{[\alpha\beta]\gamma} + N_{[\beta\alpha]\gamma} &= 0 \\ N_{[\alpha\beta]\gamma} + N_{[\beta\gamma]\alpha} + N_{[\gamma\alpha]\beta} &= 0 \end{aligned} \tag{5.10}$$

The form of (5, 9) guarantees the symmetry of $\psi_{A,B,C}$ under interchange of the pairs αp; βq; γr . Substituting (5, 9) into (5.8) and making use of (5.10) we have

$$J^{Ri} = J^{Ri}(D) + J^{Ri}(DN) + J^{Ri}(N) \tag{5.11}$$

where

$$J^{Ri}(D) = \bar{D}^{\alpha\beta\gamma,pqr}(\Gamma^R)_\alpha^{\alpha'}(T^i)_p^{p'} D_{\alpha'\beta\gamma,p'qr} \tag{5.12}$$

$$\begin{aligned} J^{Ri}(DN) = {} & 2\bar{D}^{\alpha\beta\gamma;pqr}(\Gamma^R)_\alpha^{\alpha'}(T^i)_p^{p'}\epsilon_{p'q} N_{[\alpha'\beta]\gamma,r} \\ & + 2\epsilon^{pq}\bar{N}^{[\alpha\beta]\gamma,r}(\Gamma^R)_\alpha^{\alpha'}(T^i)_p^{p'} D_{\alpha'\beta\gamma,p'qr} \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} J^{Ri}(N) = {} & \left[\bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma}\right]^i_{8X+2Y} \\ & + \left[\bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\alpha^{\alpha'} N_{[\gamma\alpha']\beta}\right]^i_{10X-2Y} \end{aligned} \tag{5.14}$$

where X and Y denote terms of the form

$$\bar{N}^p (T^i)_p^{p'} N_{p'}$$

and

$$\bar{N}^r N_r (T^i)_p^p \tag{5.15}$$

respectively.

The Breaking of $\tilde{U}(8)$.

We have seen that under the subgroup $U(2) \times \tilde{U}(4)$ the representations of $\tilde{U}(8)$ become reducible. We may further reduce the symmetry by going to the subgroup $U(2) \times \mathcal{L}_4$ where $\mathcal{L}_4$ is the homogeneous Lorentz group. Essentially only the further reduction of $\tilde{U}(4)$ to $\mathcal{L}_4$ is involved. This reduction is related to the existence of the matrix C of Appendix A which is constant under $\mathcal{L}_4$ but not under $\tilde{U}(4)$.

We have shown in Appendix A equations (A.35) and (A.36) that the matrices

$$(\gamma^\mu C^{-1})_{\alpha\beta} \quad , \quad (\sigma^{\mu\nu} C^{-1})_{\alpha\beta} \tag{5.16}$$

are symmetric, while

$$(C^{-1})_{\alpha\beta} \quad , \quad (i\gamma^\mu\gamma^5 C^{-1})_{\alpha\beta} \ , \ (\gamma^5 C^{-1})_{\alpha\beta} \tag{5.17}$$

are antisymmetric.

We may expand the tensor fields $N_{[\alpha\beta]\gamma}$, $D_{\alpha\beta\gamma}$ in terms of these matrices. The coefficients multiplying the matrices have less restraints than $N_{[\alpha\beta]\gamma}$ or $D_{\alpha\beta\gamma}$, as some of the symmetry or antisymmetry will be contained in the matrices themselves. In the next section we shall impose restraints by requiring the fields to satisfy field equations. The fields will then describe particles. Their interpretation will be more simple in terms of the expansion coefficients. Note, however, that we are retaining the full $\tilde{U}(4)$ symmetry and are not assuming $C^{\alpha\beta}$ constant. We are merely making use of its symmetry properties.

Since $D_{\alpha\beta\gamma}$ is symmetric we expand in symmetric matrices only

$$D_{\alpha\beta\gamma} = D_{\mu\alpha}(\gamma^\mu C^{-1})_{\beta\gamma} + \frac{1}{2} D_{\mu\nu\alpha}(\sigma^{\mu\nu}C^{-1})_{\beta\gamma} \quad (5.18)$$

The constraints applying to $D_{\mu\alpha}$, $D_{\mu\nu\alpha}$ are obtained by multiplying (5.18) by the antisymmetric matrices $C^{\alpha\beta}, (C i\gamma^\rho\gamma^5)^{\alpha\beta}, (C\gamma^5)^{\alpha\beta}$ which annihilate the left hand side. They yield

$$\gamma^\mu D_\mu + \frac{1}{2}\sigma^{\mu\nu} D_{\mu\nu} = 0 \quad (5.19)$$

$$\gamma^\mu\gamma^\rho\gamma^5 D_\mu + \frac{1}{2}\sigma^{\mu\nu}\gamma^\rho\gamma^5 D_{\mu\nu} = 0 \quad (5.20)$$

and

$$\gamma^\mu\gamma^5 D_\mu + \frac{1}{2}\sigma^{\mu\nu}\gamma^5 D_{\mu\nu} = 0 \quad (5.21)$$

Comparing (5.21) and (5.19) we have

$$\gamma^\mu D_\mu = 0 \quad (5.22)$$

$$\sigma^{\mu\nu} D_{\mu\nu} = 0 \quad (5.23)$$

Eliminating γ^5 from (5.20) and using (5.22) and (5.23)

$$\{\gamma^\mu, \gamma^\rho\} D_\mu - \frac{1}{2}[\sigma^{\mu\nu}, \gamma^\rho] D_{\mu\nu} = 0$$

or $$2D^\rho + i\gamma^\nu D^\rho{}_\nu - i\gamma^\mu D_\mu{}^\rho = 0$$

i.e. $$D_\rho + i\gamma^\nu D_{\rho\nu} = 0 \quad (5.24)$$

where we have made the simplifying assumption

$$D_{\nu\mu} = -D_{\mu\nu} \quad (5.25)$$

which we take as part of the definition (5.18). We note that (5.23) is contained in (5.22) and (5.24).

The field $N_{[\alpha\beta]\gamma}$ may be expanded

$$N_{[\alpha\beta]\gamma} = (C^{-1})_{\alpha\beta} K_\gamma + (\gamma^5 C^{-1})_{\alpha\beta} N_\gamma + (i\gamma^\mu\gamma^5 C^{-1})_{\alpha\beta} N_{\mu\gamma} \tag{5.26}$$

To obtain the constraints in this case we use the relation

$$N_{[\alpha\beta]\gamma} + N_{[\beta\gamma]\alpha} + N_{[\gamma\alpha]\beta} = 0 \tag{5.27}$$

Multiplying (5.27) by $C^{\alpha\beta}$ gives

$$\begin{aligned} &-4K_\gamma + K_\gamma + K_\gamma \\ &-\mathrm{tr}(\gamma^5) N_\gamma + (\gamma^5)_\gamma^{\ \alpha} N_\alpha + (\gamma^5)_\gamma^{\ \beta} N_\beta \\ &-\mathrm{tr}(i\gamma^\mu\gamma^5) N_{\mu\gamma} + (i\gamma^\mu\gamma^5)_\gamma^{\ \alpha} N_{\mu\alpha} + (i\gamma^\mu\gamma^5)_\gamma^{\ \beta} N_{\mu\beta} = 0 \end{aligned}$$

or

$$K_\gamma - (\gamma^5 N)_\gamma - (i\gamma^\mu\gamma^5 N_\mu)_\gamma = 0 \tag{5.28}$$

Multiplying (5.27) by $(C\gamma^\beta)^{\alpha\beta}$ etc. leads to no new restraint. The field equations of the next section impose in particular the restraint $K_\gamma \equiv 0$. Taking this in combination with (5.28) gives

$$N = i\gamma^\mu N_\mu \tag{5.29}$$

The 'current' $J^{Ri}(N)$ defined by (5.14) and (5.15) may now be expressed in terms of N and N_μ.

To simplify the calculation we prove the relation

$$\bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\alpha^{\alpha'} N_{[\gamma\alpha']\beta}$$

$$= \frac{1}{2} \bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\gamma^{\gamma'} N_{[\alpha\beta]\gamma'} - \bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma} \tag{5.30}$$

From equation (5.10) we have

$$N_{[\alpha\beta]\gamma} = \frac{1}{2} N_{[\alpha\beta]\gamma} + \frac{1}{2} N_{[\gamma\beta]\alpha} - \frac{1}{2} N_{[\gamma\alpha]\beta}$$

$$= \frac{1}{2}\left\{ N_{[\alpha\beta]\gamma} + N_{[\alpha\gamma]\beta} \right\} + \frac{1}{2} N_{[\gamma\beta]\alpha}$$

We have split it into symmetric and antisymmetric parts in β,γ.

Thus

$$\bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma}$$

$$= \frac{1}{4}\left\{ \bar{N}^{[\alpha\beta]\gamma} + \bar{N}^{[\alpha\gamma]\beta} \right\} (\Gamma^R)_\alpha^{\alpha'} \left\{ N_{[\alpha'\beta]\gamma} + N_{[\alpha'\gamma]\beta} \right\}$$

$$+ \frac{1}{4} \bar{N}^{[\gamma\beta]\alpha}(\Gamma^R)_\alpha^{\alpha'} N_{[\gamma\beta]\alpha'}$$

since cross terms vanish.

Multiplying out this expression and interchanging the names of the summed suffices of the various terms, we arrive at equation (5.30).

Combining (5.30) with (5.14) gives

$$J^{Ri}(N) = J_1^{Ri}(N) + J_2^{Ri}(N) \tag{5.31}$$

where
$$J_1^{Ri}(N) = \left[\bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma} \right]_{-2X+4Y}^{i} \tag{5.32}$$

and
$$J_2^{Ri}(N) = \left[\bar{N}^{[\alpha\beta]\gamma}(\Gamma^R)_\gamma^{\gamma'} N_{[\alpha\beta]\gamma'} \right]_{5X-Y}^{i} \tag{5.33}$$

Substituting (5.26) and (5.29) into (5.32) we have

$$J_1^{Ri}(N) = \Big[tr.(\Gamma^R)\,\bar{N}^{\gamma} N_{\gamma} + tr.(\Gamma^R \gamma^{\mu}\gamma^{\nu})\,\bar{N}_{\mu}{}^{\gamma} N_{\nu\gamma}$$
$$+ i\,tr.(\Gamma^R\gamma^{\mu})\{ \bar{N}_{\mu}{}^{\gamma} N_{\gamma} - \bar{N}^{\gamma} N_{\mu\gamma} \}\Big]^i_{-2x+4Y} \quad (5.34)$$

Similarly

$$J_2^{Ri}(N) = \Big[4\,\bar{N}^{\gamma}(\Gamma^R)_{\gamma}^{\gamma'} N_{\gamma'} + 4\,\bar{N}^{\mu\gamma}(\Gamma^R)_{\gamma}^{\gamma'} N_{\mu\gamma'} \Big]^i_{5x-Y} \quad (5.35)$$

These results can be further simplified by treating the cases $\Gamma^R = I$, $\Gamma^R = \gamma^{\rho}$, $\Gamma^R = \sigma^{\lambda\rho}$, $\Gamma^R = i\gamma^{\rho}\gamma^5$, $\Gamma^R = \gamma_5$ separately, so that the traces in (5.34) can be evaluated. Thus:

$$J^i(N) = \big[4\bar{N}N + 4\bar{N}^{\nu}N_{\nu} \big]^i_{-2x+4Y}$$
$$+ \big[4\bar{N}N + 4\bar{N}^{\nu}N_{\nu} \big]^i_{5x-Y} \quad (5.36)$$

$$J^{i\rho}(N) = \big[4i(\bar{N}N^{\rho} - \bar{N}^{\rho}N) \big]^i_{-2x+4Y}$$
$$+ \big[4\bar{N}\gamma^{\rho}N + 4\bar{N}^{\nu}\gamma^{\rho}N_{\nu} \big]^i_{5x-Y} \quad (5.37)$$

$$J^{i\lambda\rho}(N) = \big[4i(\bar{N}^{\lambda}N^{\rho} - \bar{N}^{\rho}N^{\lambda}) \big]^i_{-2x+4Y}$$
$$+ \big[4\bar{N}\sigma^{\lambda\rho}N + 4\bar{N}^{\nu}\sigma^{\lambda\rho}N_{\nu} \big]_{5x-Y} \quad (5.38)$$

$$J^{i\rho 5}(N) = \big[4(\bar{N}i\gamma^{\rho}\gamma^5 N + \bar{N}^{\nu}i\gamma^{\rho}\gamma^5 N_{\nu}) \big]^i_{5x-Y} \quad (5.39)$$

$$J^{i5}(N) = \left[4\left(\bar{N}\gamma^5 N + \bar{N}^\mu \gamma^5 N_\mu\right)\right]^i_{5x-Y} \tag{5.40}$$

The Field Equations*

We adopt the following wave equations for tensor fields $\psi^{\lambda\rho\ldots}_{\alpha\beta\gamma\ldots}$ of definite symmetry type:

$$(\not{p})^{\alpha'}_{\alpha} \psi^{\lambda\rho\ldots}_{\alpha'\beta\gamma\ldots} = m\,\psi^{\lambda\rho\ldots}_{\alpha\beta\gamma\ldots} \tag{5.41}$$

These equations have the advantage of being simple and general. They represent a generalisation of the approach of Bargmann and Wigner(17).

The definiteness of the symmetry type of ψ in conjunction with (5.41) implies

$$(\not{p})^{\beta'}_{\beta} \psi^{\lambda\rho\ldots}_{\alpha\beta'\gamma\ldots} = m\,\psi^{\lambda\rho\ldots}_{\alpha\beta\gamma\ldots}$$

and (5.42)

$$\left(\psi^{\lambda'\rho\ldots}_{\alpha\beta\gamma\ldots}\right)(\not{p})^{\lambda}_{\lambda'} = \left(\psi^{\lambda\rho\ldots}_{\alpha\beta\gamma\ldots}\right)m$$

etc.

This is obvious for the special case when ψ is $D_{\alpha\beta\gamma}$. We illustrate the result for $N_{[\alpha\beta]\gamma}$. Thus

$$(\not{p})^{\alpha'}_{\alpha} \times N_{[\alpha'\beta]\gamma} = m\,N_{[\alpha\beta]\gamma}$$

clearly implies

$$(\not{p})^{\beta'}_{\beta} N_{[\alpha\beta']\gamma} = m\,N_{[\alpha\beta]\gamma}$$

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* This approach was introduced in reference (3) to discuss $\widetilde{U}(12)$ theory.

Hence

$$(\not{p})_{\gamma}^{\gamma'} N_{[\alpha\beta]\gamma'}$$

$$= (\not{p})_{\gamma}^{\gamma'} (-N_{[\beta\gamma']\alpha} - N_{[\gamma'\alpha]\beta})$$

$$= m(-N_{[\beta\gamma]\alpha} - N_{[\gamma\alpha]\beta})$$

$$= m N_{[\alpha\beta]\gamma}$$

Similarly from

$$(\not{p})_{\alpha}^{\alpha'} \phi_{\alpha'}^{\beta} = \mu \phi_{\alpha}^{\beta}$$

and

$$\phi_{\beta}^{\alpha} = (A^{-1})_{\beta\beta'} \phi_{\alpha'}^{\beta'} A^{\alpha'\alpha}$$

(c.f. Equation (A.34)). We have

$$(A^{-1})_{\rho\beta} \{(\not{p})_{\alpha}^{\delta} (A^{-1})_{\lambda\delta} A^{\alpha'\lambda}\} \phi_{\alpha'}^{\beta} A^{\alpha\sigma} = \mu (A^{-1})_{\rho\beta} \phi_{\alpha}^{\beta} A^{\alpha\sigma}$$

or

$$\{(A^{-1})_{\lambda\delta} (\not{p})_{\alpha}^{\delta} A^{\alpha\sigma}\}\{(A^{-1})_{\rho\beta} \phi_{\alpha'}^{\beta} A^{\alpha'\lambda}\} = \mu \phi_{\rho}^{\sigma}$$

i.e.

$$\phi_{\rho}^{\lambda} (\not{p})_{\lambda}^{\sigma} = \phi_{\rho}^{\sigma} \cdot \mu$$

as required.

The wave function $\psi_{\alpha\beta\gamma\ldots}^{\lambda\rho\ldots}$ may be thought of as being obtained from fundamental spinors ψ_a satisfying the Dirac equation

$$(\not{p})_{\alpha}^{\beta} \psi_{\beta} = m \psi_{\alpha} \qquad (5.43)$$

This equation is invariant under the inhomogeneous Lorentz group. Under the homogeneous Lorentz group the four component spinor ψ_a

separates into two 2-component spinors. We may regard $\mathcal{L}_4$ as $SU_L(2) \times SU_R(2)$. Under this group the 4-component spinor decomposes thus: $4 = (2, 1) + (1, 2)$.

The Dirac equation, however, relates these subgroups.

ψ_α is a representation of the inhomogeneous Lorentz group. We may think of this group as obtained from an inhomogeneous generalisation of $\widetilde{U}(4)$; the reduction being brought about by the application of wave equations. Since the Dirac spinor was obtained from the 4-dimensional representation of inhomogeneous $\widetilde{U}(4)$, it is evident that tensor fields formed from products of such spinors will also be representations of this group. We note incidentally that the dimensions of these representations and their product decompositions are identical with those of $\widetilde{U}(4)$. We may anticipate that the reduction to the inhomogeneous Lorentz group will make these decompositions further reducible. These in turn will decompose under the homogeneous Lorentz group. Specific examples of this situation will be seen in our treatment of $D_{\alpha\beta\gamma}$, $N_{[\alpha\beta]\gamma}$ and Φ^β_α.

<u>Reduction of the Totally Symmetric Tensor.</u>

In accordance with (5.41) $D_{\alpha\beta\gamma}$ is to satisfy the equation

$$(\not p)_\alpha^{\alpha'} D_{\alpha'\beta\gamma} = m D_{\alpha\beta\gamma} \tag{5.44}$$

Using (5.18), this implies

$$(\not p - m I)_\alpha^{\alpha'} D_{\nu\lambda'} = 0 \tag{5.45}$$

and

$$(\not p - m I)_\alpha^{\alpha'} D_{\nu\gamma\alpha'} = 0 \tag{5.46}$$

The equation

$$(\not{p})_\beta^{\beta'} D_{\alpha\beta'\gamma} = m D_{\alpha\beta\gamma}$$

gives

$$p_\rho \left(D_{\nu\alpha} (\gamma^\rho \gamma^\nu C^{-1})_{\beta\gamma} + \tfrac{1}{2} D_{\nu\tau\alpha} (\gamma^\rho \sigma^{\nu\tau} C^{-1})_{\beta\gamma} \right)$$

$$= m \left(D_{\nu\alpha} (\gamma^\nu C^{-1})_{\beta\gamma} + \tfrac{1}{2} D_{\nu\tau\alpha} (\sigma^{\nu\tau} C^{-1})_{\beta\gamma} \right)$$

Multiplying this by $C^{\gamma\beta}$ gives

$$4 p_\rho D^\rho{}_\alpha = 0 \tag{5.47}$$

by $(C\gamma_\lambda)^{\gamma\beta}$ gives

$$2i(p^\nu D_{\nu\lambda} - p^\tau D_{\lambda\tau})_\alpha = 4 m D_{\lambda\alpha}$$

or, from (5.25)

$$p^\lambda D_{\lambda\nu\alpha} = -im D_{\nu\alpha} \tag{5.48}$$

Note (5.47) is contained in (5.48) and (5.25).
Multiplying by $(C\sigma^{\lambda\kappa})^{\gamma\beta}$ gives

$$4i \{ p^\kappa D^\lambda{}_\alpha - p^\lambda D^\kappa{}_\alpha \} = -2m \{ D^{\kappa\lambda} - D^{\lambda\kappa} \}_\alpha$$

or, using (5.25)

$$p_\nu D_{\tau\alpha} - p_\tau D_{\nu\alpha} = im D_{\nu\tau\alpha} \tag{5.49}$$

No other conditions can be derived.

Reduction of the Mixed Symmetry Tensor.

The tensor $N_{[\alpha\beta]\gamma}$ satisfies the equivalent equations

$$(\not{p})_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma} = m N_{[\alpha\beta]\gamma}$$

and $$(\not{p})_\gamma^{\gamma'} N_{[\alpha\beta]\gamma'} = m N_{[\alpha\beta]\gamma} \tag{5.50}$$

Substituting the expansion (5.26) in the latter of these equations leads to

$$(\not{p} - m)K = (\not{p} - m)N = (\not{p} - m)N_\nu = 0 \tag{5.51}$$

Substituting (5.26) in the former gives: (a) on multiplication by $C^{\beta\alpha}$

$$0 = 4mK_\gamma$$

or

$$K_\gamma = 0 \tag{5.52}$$

(b) on multiplication by $(C\gamma^5)^{\beta\alpha}$

$$-4i\ p^\mu N_{\mu\gamma} = -4m\ N_\gamma$$

or

$$p^\mu N_{\mu\gamma} = -im\ N_\gamma \tag{5.53}$$

(c) on multiplication by $(Ci\gamma^\lambda\gamma^5)^{\beta\alpha}$

$$4i\ p^\lambda N_\gamma = -4m\ N^\lambda{}_\gamma$$

or

$$p_\lambda N_\gamma = im\ N_{\lambda\gamma} \tag{5.54}$$

(d) on multiplication by $(C\gamma^5\sigma^{\lambda\kappa})^{\beta\alpha}$

$$-p_\rho i\ tr.(\gamma^\rho\gamma^\mu \sigma^{\lambda\kappa})\ N_{\mu\gamma} = 0$$

or

$$p_\mu N_{\nu\alpha} - p_\nu N_{\mu\alpha} = 0 \tag{5.55}$$

Multiplication by $(C\gamma^{\lambda})^{\beta\alpha}$ leads to

$$p^{\lambda} K_{\gamma} = 0$$

which is contained in $K_{\gamma} = 0$.

Reduction of the 'meson' field Φ_{α}^{β}

Φ_{a}^{β} has the expansion

$$\Phi_{\alpha}^{\beta} = \left[\phi I + \phi_{\mu}\gamma^{\mu} + \tfrac{1}{2}\phi_{\mu\nu}\sigma^{\mu\nu} + \phi_5\gamma^5 + \phi_{\mu 5}\, i\gamma^{\mu}\gamma^5\right]_{\alpha}^{\beta} \tag{5.56}$$

Substituting (5.56) in the wave equation

$$(\not p)_{\alpha}^{\alpha'} \Phi_{\alpha'}^{\beta} = \mu\, \Phi_{\alpha}^{\beta} \tag{5.57}$$

we have

(a) on multiplication by $\frac{1}{4}\delta_{\beta}^{\alpha}$

$$p^{\mu}\phi_{\mu} = \mu\phi \tag{5.58}$$

(b) on multiplication by $\frac{1}{4}(\gamma^{\lambda})_{\beta}^{\alpha}$

$$p^{\lambda}\phi + \frac{i}{2}\left\{p^{\mu}\phi_{\mu}{}^{\lambda} - p^{\nu}\phi^{\lambda}{}_{\nu}\right\} = \mu\phi^{\lambda}$$

or, assuming $\quad \phi_{\mu\nu} = -\phi_{\nu\mu}$ $\hfill$ (5.59)

$$p^{\mu}\phi_{\mu\lambda} = -i\mu\phi_{\lambda} + i p_{\lambda}\phi \tag{5.60}$$

(c) on multiplication by $\quad \frac{1}{4}(\sigma^{\lambda\kappa})_{\beta}^{\alpha}$

$$i\{p^{\varkappa}\phi^{\lambda} - p^{\lambda}\phi^{\varkappa}\} = -\frac{\mu}{2}\{\phi^{\varkappa\lambda} - \phi^{\lambda\varkappa}\}$$

or

$$p_\mu \phi_\nu - p_\nu \phi_\mu = i\mu\phi_{\mu\nu} \tag{5.61}$$

(d) on multiplication by $\frac{1}{4}(\gamma^5)^\alpha_\beta$

$$-i\, p^\mu \phi_{\mu 5} = -\mu\phi_5$$

or

$$p^\mu \phi_{\mu 5} = -i\mu\phi_5 \tag{5.62}$$

(e) on multiplication by $\frac{1}{4}(i\gamma^\lambda\gamma^5)^\alpha_\beta$

$$i\, p^\lambda \phi_5 = -\mu\phi^\lambda{}_5$$

or

$$p_\mu \phi_5 = i\mu\phi_{\mu 5} \tag{5.63}$$

If we introduce

$$\phi'_\mu = \phi_\mu - \frac{p_\mu}{\mu}\phi$$

(5.58) and (5.60) give rise to

$$\left.\begin{aligned} p^\mu\phi'_\mu &= 0 \\ p^\nu\phi_{\mu\nu} &= -i\mu\phi'_\lambda \\ (p^\mu p_\mu - \mu^2)\phi &= 0 \end{aligned}\right\} \tag{5.64}$$

In the above, the inclusion of $\phi\,\delta_\alpha^\beta$ in Φ_α^β violated our requirement that Φ_α^β be an irreducible representation of $\tilde{U}(4)$ formed from a quark-antiquark field.

We should in fact set

$$\phi = 0 \tag{5.65}$$

Interpretation of the Field Equations

From the field equations above we see by comparison with the Kemmer theory(18) of spin 0 and spin 1 particles that

$$\phi_{\mu 5} \quad \text{and} \quad \phi_5$$

together describe a spin 0 particle, while

$$\phi_{\mu\nu} \quad \text{and} \quad \phi_\mu$$

describe a spin 1 particle.

Similarly we see that

$$D_{\mu\nu\alpha} \quad \text{and} \quad D_{\mu\alpha}$$

satisfy the Rarita-Schwinger(19) field equations of a spin $^3/_2$ particle and

$$N_{\mu\alpha}, \quad N_\alpha$$

reduce to the description of a spin ½ particle.

The field equations are a consequence of $\tilde{U}(8)$ field equations

$$(\not p)_A^{\ A'} \Phi_{A'}^{\ B} = \mathcal{N}\, \Phi_A^{\ B} \tag{5.66}$$

and

$$(\not p)_A^{\ A'} \psi_{A'BC} = m\, \psi_{ABC} \tag{5.57}$$

where

$$(\not p)_A^{\ B} = (\not p)_\alpha^{\ \beta}\, \delta_p^{\ q} \tag{5.68}$$

These equations are not invariant under $\tilde{U}(8)$ and may be regarded as breaking the symmetry.

We see from these equations why $D_{\alpha\beta\gamma}$ and $N_{[\alpha\beta]\gamma}$ were assigned the same mass. Since, the broken symmetry can be obtained by first going to the $U(3) \times \tilde{U}(4)$ subgroup and then breaking $\tilde{U}(4)$ by wave equations, it is not clear that anything much would be lost by having different masses. We shall see below that ϕ_{α}^{β} describe two kinds of particles whose mass splitting is greater than that between $D_{\alpha\beta\gamma}$ and $N_{[\alpha\beta]\gamma}$. Thus the point of giving different masses seems lost unless we can further reduce the symmetry by reducing $\tilde{U}(4)$ in such a way that the natural group for particles arises, before the wave equations are imposed on the system.

Such a reduction is brought about by the subgroup $\tilde{Sp}(4)$ of $\tilde{U}(4)$ discussed in the next Chapter. As we shall see, this requires a modification in the treatment of the wave equations, but the results differ only in the assignment of masses.

There is one respect in which this proposed modification is unsatisfactory; that is, in the limit of zero momentum $\tilde{U}(8)$ and $\tilde{U}(12)$ lead to a $U(4)$ and $U(6)$ classification of particles. The proposed modification allows diverse masses to be assigned within a given $U(4)$ or $U(6)$ multiplet.

This result will be discussed in more detail in the next chapter.

CHAPTER 6

PREDICTIONS OF $\tilde{U}(8)$ FOR EXACT AND BROKEN SYMMETRY

In this chapter we consider the modification of the field equations resulting from the assumption that a separate Bargmann-Wigner equation is satisfied by each irreducible representation of $\tilde{Sp}(4)$ rather than each irreducible representation of $\tilde{U}(4)$. The results of these considerations are incorporated in the $\tilde{U}(8)$ theory from which we derive some predictions by considering the coupling of the photon to the baryon structure.

The Group $\tilde{Sp}(4)$.

The group $\tilde{Sp}(4)$ is the subgroup of $\tilde{U}(4)$ with generators γ^{μ} and $\sigma^{\mu\nu}$. It has the following properties

(a) it contains the Lorentz group;

(b) it is simple;

(c) it leaves form invariant an antisymmetric second rank tensor which may be taken as a metric;

(d) it is the smallest simple subgroup of $U(4)$ containing the Lorentz group.

We wish to establish the existence of a one-to-one correspondence between the irreducible representations of $\tilde{Sp}(4)$ and the inhomogeneous Lorentz group.

Consider the Dirac equation

$$(\not{p} - m)\psi = 0 \tag{6.1}$$

for a particle at rest (6.1) becomes

$$(\gamma_0 - I)\psi = 0 \tag{6.2}$$

$$(\gamma_0 - I)\psi = 0 \qquad (6.2)$$

With γ_0 as defined in Appendix A, (6.2) implies

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{bmatrix} \qquad (6.3)$$

Equation (6.2) is invariant under the group of spatial rotations; ψ_1 and ψ_2 forming the two components of the spinor field of SU(2).

The general spinor ψ satisfying (6.1) can be obtained from (6.3) by operating on it with the general transformation of the inhomogeneous Lorentz group.

If we form from a set of spinors $\psi^{(1)}$, $\psi^{(2)}$,...., satisfying (6.2), the various tensors of definite symmetry, the following situation results.

Only Young's tableaux with one or two rows give rise to a non-vanishing tensor field. This is because the spinors ψ are effectively SU(2) spinors.

We can transform from this situation by operating on these tensor fields with the general transformation of the inhomogeneous Lorentz group. The resulting tensor field can be considered as the reduction of $\tilde{U}(4)$. tensors with Bargmann-Wigner equations. The $\tilde{U}(4)$ tensors have the 'same' Young's tableaux as the corresponding SU(2) tensors.

Tensors with more than 2 rows are annihilated by the Bargmann-Wigner equations, as is evident from the consideration of their implication in the next frame.

The reason for choosing $\tilde{U}(4)$ above is to keep the one-to-one correspondence between the Young's tableaux.

The changes that arise from choosing $\tilde{Sp}(4)$ instead of $\tilde{U}(4)$

are the following: suppose we have a tensor field of $\tilde{U}(4)$ with an associated two-rowed tableau, this will be reducible under $\tilde{Sp}(4)$. Imposition of the Bargmann-Wigner equations annihilates all but one of these $\tilde{Sp}(4)$ representations, and gives rise to the description of a particle of spin corresponding to the SU(2) interpretation of the tableaux for the remaining one.

That this is the case follows from the fact that the inhomogeneous generalisations of $\tilde{Sp}(4)$ contains the inhomogeneous Lorentz group and the fact that only one particle was described by the parent $\tilde{U}(4)$ tensor.

So far the advantage of using $\tilde{Sp}(4)$ rather than $\tilde{U}(4)$ has not shown itself. It becomes useful only when antiparticles as well as particles enter into the description. This is because the tensor $\bar{\Psi}$ in the rest frame has its non zero components in the 'wrong' place - a situation which can be rectified with the metric tensor $C^{\alpha\beta}$. Here $\tilde{U}(4)$ fails us, while $\tilde{Sp}(4)$ remains satisfactory. The one-to-one correspondence is thus maintained by choosing those representations of $\tilde{Sp}(4)$ which do not vanish unless operated on by the Bargmann-Wigner equations.

We examine below the detailed situation for the tensor fields $D_{\alpha\beta\gamma}$, $\bar{\Phi}^{\beta}_{\alpha}$, $N_{[\alpha\beta]\gamma}$ of $\tilde{U}(4)$ when reduced under $\tilde{Sp}(4)$ symmetry.

To reduce the tensor $D_{\alpha\beta\gamma}$ under $\tilde{Sp}(4)$ we must remove all traces with respect to the metric $C^{\alpha\beta}$. Since, however, $D_{\alpha\beta\gamma}$ is symmetric these traces are zero, i.e.

$$C^{\alpha\beta} D_{\alpha\beta\gamma} = C^{\beta\gamma} D_{\alpha\beta\gamma} = C^{\gamma\alpha} D_{\alpha\beta\gamma} = 0 \tag{6.4}$$

so that $D_{\alpha\beta\gamma}$, like all completely symmetrised tensors, is already an irreducible tensor of $\tilde{Sp}(4)$. The treatment of $D_{\alpha\beta\gamma}$ under the field equations is thus unchanged.

Φ_α^β has the expansion

$$\Phi_\alpha^\beta = \left\{ \phi_5 \gamma^5 + \phi_{\mu 5} (i \gamma^\mu \gamma^5) \right\}_\alpha^\beta + \left\{ \phi_\mu \gamma^\mu + \frac{1}{2} \phi_{\mu\nu} \sigma^{\mu\nu} \right\}_\alpha^\beta$$

$$= \Phi_\alpha^{(1)\beta} + \Phi_\alpha^{(2)\beta} \qquad (6.5)$$

$\Phi_\alpha^{(1)\beta}$ and $\Phi_\alpha^{(2)\beta}$ are separate irreducible representations of $\widetilde{Sp}(4)$. To see this we note that the commutators of $\Phi_\alpha^{(1)\beta}$ or $\Phi_\alpha^{(2)\beta}$ with the generators $\gamma^\mu, \sigma^{\mu\nu}$ of $\widetilde{Sp}(4)$ give rise to tensors of the same form.

The wave equation

$$(\not{p})_\alpha^{\alpha'} \Phi_{\alpha'}^\beta = \mu \Phi_\alpha^\beta$$

can now be replaced by the two eauations

$$(\not{p})_\alpha^{\alpha'} \Phi_{\alpha'}^{(1)\beta} = \mu^{(1)} \Phi_\alpha^{(1)\beta}$$

and (6.6)

$$(\not{p})_\alpha^{\alpha'} \Phi_{\alpha'}^{(2)\beta} = \mu^{(2)} \Phi_\alpha^{(2)\beta}$$

Defining

$$\Phi_{\alpha\beta}^{(1)} = \Phi_\alpha^{(1)\beta'} (C^{-1})_{\beta'\beta}$$

and (6.7)

$$\Phi_{\alpha\beta}^{(2)} = \Phi_\alpha^{(2)\beta'} (C^{-1})_{\beta'\beta}$$

we see $\Phi_{\alpha\beta}^{(1)}$ is antisymmetric in α,β while $\Phi_{\alpha\beta}^{(2)}$ is symmetric in α,β.

Equations (6.6) now read

$$(\not{p})_\alpha^{\alpha'} \bar{\Phi}^{(1)}_{\alpha'\beta} = \mathcal{N}^{(1)} \bar{\Phi}^{(1)}_{\alpha\beta}$$

and (6.8)

$$(\not{p})_\alpha^{\alpha'} \bar{\Phi}^{(2)}_{\alpha'\beta} = \mathcal{N}^{(2)} \bar{\Phi}^{(2)}_{\alpha\beta}$$

while the equivalent equations

$$(\not{p})_{\beta'}^{\beta} \bar{\Phi}^{(1)\beta'}_{\alpha} = \mathcal{N}^{(1)} \bar{\Phi}^{(1)\beta}_{\alpha}$$

$$(\not{p})_{\beta'}^{\beta} \bar{\Phi}^{(2)\beta'}_{\alpha} = \mathcal{N}^{(2)} \bar{\Phi}^{(2)\beta}_{\alpha} \qquad (6.9)$$

are easily shown to give the same result (6.8). We have thus exhibited the value of the C-matrix in reducing $\bar{\Phi}_\alpha^\beta$ to a two-rowed representation.

The reduction of (6.6) can be carried out as in the last chapters, the results differing only in different masses.being assigned to the spin 0 ($\bar{\Phi}^{(1)\beta}_{\alpha}$) and spin 1 ($\bar{\Phi}^{(2)\beta}_{\alpha}$) particle fields.

The analysis of the field $N_{[\alpha\beta]\gamma}$ is more complex.

We first contract out the traces of $N_{[\alpha\beta]\gamma}$ with respect to $C^{\alpha\beta}$. That is we put

$$N_{[\alpha\beta]\gamma} = N^{(T)}_{[\alpha\beta]\gamma} + (C^{-1})_{\alpha\beta} H^{(1)}_\gamma + (C^{-1})_{\beta\gamma} H^{(2)}_\alpha + (C^{-1})_{\gamma\alpha} H^{(3)}_\beta \qquad (6.10)$$

where $N^{(T)}_{[\alpha\beta]\gamma}$ is traceless.

Recalling the relation of the last chapter

$$N_{[\alpha\beta]\gamma} = (C^{-1})_{\alpha\beta} K_\gamma + (\gamma^5 C^{-1})_{\alpha\beta} N_\gamma + (i\gamma^\nu\gamma^5 C^{-1})_{\alpha\beta} N_{\nu\gamma} \qquad (6.11)$$

with its symmetry constraint

$$K_\gamma - (\gamma^5 N)_\gamma - (i\gamma^\nu\gamma^5 N_\nu)_\gamma = 0 \tag{6.12}$$

we obtain from (6.10), by contraction with $C^{\alpha\beta}$, $C^{\beta\gamma}$, $C^{\gamma\alpha}$, the relations

$$4K_\gamma = 4H^{(1)}_\gamma - H^{(2)}_\gamma - H^{(3)}_\gamma$$

$$2K_\gamma = H^{(1)}_\gamma - 4H^{(2)}_\gamma + H^{(3)}_\gamma$$

$$2K_\gamma = H^{(1)}_\gamma + H^{(2)}_\gamma - 4H^{(3)}_\gamma$$

where use has been made of (6.12). Solving these equations, we have

$$H^{(1)}_\gamma = \frac{4}{5}K_\gamma$$

and

$$H^{(2)}_\gamma = H^{(3)}_\gamma = -\frac{2}{5}K_\gamma \tag{6.13}$$

Thus putting

$$K_{[\alpha\beta]\gamma} = \frac{2}{5}\left\{(C^{-1})_{\alpha\beta}2K_\gamma - (C^{-1})_{\beta\gamma}K_\alpha - (C^{-1})_{\gamma\alpha}K_\beta\right\} \tag{6.14}$$

we have

$$N_{[\alpha\beta]\gamma} = N^{(T)}_{[\alpha\beta]\gamma} + K_{[\alpha\beta]\gamma} \tag{6.15}$$

Both $N^{(T)}_{[\alpha\beta]\gamma}$ and $K_{[\alpha\beta]\gamma}$ are irreducible tensors of $\widetilde{Sp}(4)$.

$K_{[\alpha\beta]\gamma}$ can be expanded in analogy with equation (6.11), thus:

$$\frac{5}{2}K_{[\alpha\beta]\gamma} = (C^{-1})_{\alpha\beta}L_\gamma + (\gamma^5 C^{-1})_{\alpha\beta}M_\gamma + (i\gamma^\nu\gamma^5 C^{-1})_{\alpha\beta}M_{\nu\gamma} \tag{6.16}$$

Multiplying (6.14) and (6.16) by $C^{\alpha\beta}$, $(C\gamma^5)^{\alpha\beta}$ and $(Ci\gamma^\lambda\gamma^5)^{\alpha\beta}$ and comparing, we discover

$$L_\gamma = \frac{5}{2} K_\gamma$$

$$M_\gamma = -\frac{1}{2}(\gamma^5 K)_\gamma \qquad (6.17)$$

$$M_{\lambda\gamma} = -\frac{1}{2}(i\gamma_\lambda \gamma^5 K)_\gamma$$

or

$$K_{[\alpha\beta]\gamma} = (C^{-1})_{\alpha\beta} K_\gamma - (\gamma^5 C^{-1})_{\alpha\beta} \frac{1}{5}(\gamma^5 K)_\gamma$$

$$- \frac{1}{2}(i\gamma^\nu \gamma^5)_{\alpha\beta} (i\gamma_\nu \gamma^5 K)_\gamma \qquad (6.18)$$

Thus (6.11), (6.15) and (6.18) yield

$$N^{(T)}_{[\alpha\beta]\gamma} = (\gamma^5 C^{-1})_{\alpha\beta} N'_\gamma + (i\gamma^\nu \gamma^5) N'_{\nu\gamma} \qquad (6.19)$$

where

$$N'_\gamma = N_\gamma - \frac{1}{5}(\gamma^5 K)_\gamma$$

and

$$N'_{\nu\gamma} = N_{\nu\gamma} - \frac{1}{2}(i\gamma_\nu \gamma^5 K)_\gamma \qquad (6.20)$$

The symmetry restraint (6.11) is equivalent to

$$(\gamma^5 N')_\gamma + (i\gamma^\nu \gamma^5 N'_\nu)_\gamma = 0 \qquad (6.21)$$

as is easily checked by putting (6.20) back into (6.21) and comparing with (6.11).

The field equation

$$(\not{p})_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma} = m N_{[\alpha\beta]\gamma}$$

can now be replaced by

$$(\not{p})_\alpha^{\alpha'} N^{(T)}_{[\alpha'\beta]\gamma} = M N^{(T)}_{[\alpha\beta]\gamma} \qquad (6.22)$$

and

$$(\not{p})_\alpha^{\alpha'} K_{[\alpha'\beta]\gamma} = \kappa K_{[\alpha\beta]\gamma} \qquad (6.23)$$

Both $N^{(T)}_{[\alpha'\beta]\gamma}$ and $K_{[\alpha\beta]\gamma}$ are tensors of symmetry type (2, 1) and are comparable with $N_{[\alpha\beta]\gamma}$. The field equations are, as a result, of the same form. The only difference lies in the extra relations existing between the coefficients.

Thus one field relation of (6.23) is $L_\gamma = 0$ and from (6.17) we immediately deduce

$$M_\gamma = M_{\mu\gamma} = 0$$

or

$$K_{[\alpha\beta]\gamma} = 0$$

The fact that there is no term $K_\gamma (C^{-1})_{\alpha\beta}$ in the expansion (6.19) makes no difference, since the field equations would destroy this anyway. The final result then differs from the $\tilde{U}(4)$ case only in the replacement of N_γ and $N_{\mu\gamma}$ by N_γ' and $N_{\mu\gamma}'$ which is merely a matter of name, and we may drop the prime without confusion. We return now to the development of $\tilde{U}(8)$ theory and will show how to introduce the above modification later. Now that the field equations are postulated, physical meaning is given to the currents J^{Ri}. Some economy in terms results from the field equations; thus from equation (5.54) we have

or $$N_\nu = -\left(\frac{i p_\nu}{m}\right) N \tag{6.24}$$

If we denote incoming and outgoing momenta by p' and p and put

$$P = p + p' \; ; \quad Q = p - p' \; ; \quad r_\rho = \epsilon_{\rho\nu\kappa\lambda} P^\nu Q^\kappa \gamma^\lambda \gamma^5 \tag{6.25}$$

we have a convenient set of variables in terms of which J^{iR} may be expressed.

We note the relations

$$P^2 + Q^2 = 4m^2$$
$$P^2 - Q^2 = 4p^\mu p'_\mu$$

and $$2P^2 = 4m^2 + 4p^\mu p'_\mu$$

Comparing this last relation with (5.36) we have (taking into account (6.24))

$$J^i(N) = \frac{2P^2}{m^2}[\bar{N}N]^i_{3X+3Y} \tag{6.26}$$

Consider the expression

$$4\bar{N}\gamma^\rho N + 4\bar{N}^\mu\gamma^\rho N_\mu$$

This may be written

$$2\frac{p_\mu p'_\nu}{m^2}\bar{N}\,2\gamma^\mu\gamma^\rho\gamma^\nu + (\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\nu\gamma^\mu\gamma^\rho)N$$

where we have used (6.24) and (5.51), or more symmetrically as

$$\frac{p_\mu p'_\nu}{m^2}\bar{N}\,4\gamma^\mu\gamma^\rho\gamma^\nu + (\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\nu\gamma^\mu\gamma^\rho) + (\gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu)N$$

Putting $$4\gamma^\mu\gamma^\rho\gamma^\nu = -2\gamma^\mu\gamma^\nu\gamma^\rho + 4\gamma^\mu g^{\rho\nu} - 2\gamma^\rho\gamma^\mu\gamma^\nu + 4g^{\rho\mu}\gamma^\nu$$

this becomes

$$\frac{p_\mu p'_\nu}{m^2}\bar{N}\{\gamma^\rho,[\gamma^\nu,\gamma^\mu]\} + 4\gamma^\mu g^{\rho\nu} + 4\gamma^\nu g^{\rho\mu}N$$
$$= -\frac{4p_\mu p'_\nu}{m^2}\bar{N}\epsilon^{\rho\nu\mu\kappa}\gamma_\kappa\gamma^5 N + \frac{4P^\rho}{m}\bar{N}N$$

But

$$r_\rho = \epsilon_{\rho\nu\mu\kappa}P^\nu Q^\mu\gamma^\kappa\gamma^5$$
$$= \epsilon_{\rho\nu\mu\kappa}(p^\nu p^\mu - p^\nu p'^\mu + p'^\nu p^\mu - p'^\nu p'^\mu)\gamma^\kappa\gamma^5$$
$$= 2\epsilon_{\rho\nu\mu\kappa}p^\mu p'^\nu\gamma^\kappa\gamma^5$$

Hence we have

$$4\,\bar{N}\gamma^{\rho}N + 4\,\bar{N}^{\nu}\gamma^{\rho}N_{\nu}$$

$$= -\frac{2}{m^2}\bar{N}\Gamma^{\rho}N + \frac{4P^{\rho}}{m}\bar{N}N \tag{6.27}$$

Comparing this relation with (5.37) we easily deduce

$$J^{i\rho}(N) = -\frac{2}{m^2}\left[\bar{N}\Gamma^{\rho}N\right]^{i}_{5x-y} + \frac{4P^{\rho}}{m}\left[\bar{N}N\right]^{i}_{3x+3y} \tag{6.28}$$

(5.38), (5.39) and (5.40) can be re-expressed using (6.24), as

$$J^{i\lambda\rho}(N) = \frac{2P^2}{m^2}\left[\bar{N}\sigma^{\lambda\rho}N\right]^{i}_{5x-y} + 4i(p^{\rho}p'^{\lambda} - p^{\lambda}p'^{\rho})\left[\bar{N}N\right]^{i}_{-2x+4y} \tag{6.29}$$

$$J^{i\rho 5}(N) = \frac{2P^2}{m^2}\left[\bar{N}\,i\gamma^{\rho}\gamma^{5}N\right]^{i}_{5x-y} \tag{6.30}$$

$$J^{i5}(N) = \frac{2P^2}{m^2}\left[\bar{N}\gamma^{5}N\right]^{i}_{5x-y} \tag{6.31}$$

To calculate $J^{Ri}(D)$ and $J^{Ri}(D,N)$, defined by equations (5.12) and (5.13), combine the expansions (5.18) and (5.26) with the field equations to give

$$D_{\alpha\beta\gamma} = D_{\nu\alpha}(\gamma^{\nu}C^{-1})_{\beta\gamma} + \frac{p_{\nu}}{im}D_{\tau\alpha}(\sigma^{\nu\tau}C^{-1})_{\beta\gamma} \tag{6.32}$$

and

$$N_{[\alpha\beta]\gamma} = (\gamma^{5}C^{-1})_{\alpha\beta}N_{\gamma} + \left(-\frac{ip_{\nu}}{m}\right)(i\gamma^{\nu}\gamma^{5}C^{-1})_{\alpha\beta}N_{\gamma} \tag{6.33}$$

Substituting (6.32) into (5.12)

$$J^{Ri}(D) = 4\,[\bar{D}^{\nu\alpha}(\Gamma^R)_\alpha^{\ \alpha'} D_{\nu\alpha'}]^i$$

$$+ \frac{p_\mu p'_\lambda}{m^2}\, 4[\bar{D}_\nu^{\ \alpha}(\Gamma^R)_\alpha^{\ \alpha'} D_{\kappa\alpha'}]^i\{g^{\lambda\mu}g^{\kappa\nu} - g^{\kappa\mu}g^{\lambda\nu}\}$$

$$= 4\,[\bar{D}^{\nu\alpha}(\Gamma^R)_\alpha^{\ \alpha'} D_{\nu\alpha'}]^i$$

$$+ \frac{p_\nu p'^\nu}{m^2}[4\,\bar{D}_\nu^{\ \alpha}(\Gamma^R)_\alpha^{\ \alpha'} D^\nu_{\ \alpha'}]^i$$

$$- 4\,\frac{p_\nu p'^\nu}{m^2}[\bar{D}_\nu^{\ \alpha}(\Gamma^R)_\alpha^{\ \alpha'} D^\nu_{\ \alpha'}]^i$$

$$= \frac{4P^2}{m^2}[\bar{D}^{\nu\alpha}(\Gamma^R)_\alpha^{\ \alpha'} D_{\nu\alpha'}]^i$$

$$+ \frac{4}{m^2}[Q^\nu \bar{D}_\nu^{\ \alpha}(\Gamma^R)_\alpha^{\ \alpha'} Q^\nu D_{\nu\alpha'}]^i \qquad (6.34)$$

where we have used the field equation (5.47). Substituting (6.32) and (6.33) in (5.13) and using the symmetry of $D_{\alpha\beta\gamma}$

$$J^{Ri}(DN) = \Big[\{\bar{D}_\nu^{\ \gamma}(C\gamma^\nu)^{\alpha\beta} + \frac{iP_\nu}{m}\bar{D}_\nu^{\ \gamma}(C\sigma^{\nu\nu})^{\alpha\beta}\}$$

$$\times (\Gamma^R)_\alpha^{\ \alpha'} \times$$

$$\{(\gamma^5 C^{-1})_{\alpha'\beta} N_\gamma + (-\frac{iP'_\lambda}{m})(i\gamma^\lambda\gamma^5 C^{-1})_{\alpha'\beta} N_\gamma\}\Big]^i$$

+ similar expression in D and $\bar{N}$

$$= \Big[\bar{D}_\nu N \,\mathrm{tr}\Big\{ \Gamma^R \gamma^5 \gamma^\mu - i \frac{p'_\lambda}{m} (i \Gamma^R \gamma^\lambda \gamma^5 \gamma^\mu)$$

$$+ \frac{i p_\nu}{m} (\Gamma^R \gamma^5 \sigma^{\mu\nu})$$

$$+ \frac{p_\nu p'_\lambda}{m^2} (i \Gamma^R \gamma^\lambda \gamma^5 \sigma^{\nu\mu}) \Big\} \Big]^i$$

+ similar term.

Evaluating $J^{Ri}(D\ N)$ separately for the special cases,

$$J^i(DN) = 0 \tag{6.35}$$

$$J^{i\rho}(DN) = -4 \frac{p_\nu p'_\lambda}{m^2} \epsilon^{\rho\lambda\nu\mu} [\bar{D}_\mu N]^i + \text{similar term}$$

$$= 2 \epsilon^{\rho\lambda\nu\mu} P_\nu Q_\lambda [\bar{D}_\mu N - \bar{N} D_\mu]^i \tag{6.36}$$

$$J^{i\lambda\rho}(DN) = \Big\{ \frac{i p'_\nu}{m} \mathrm{tr.} (i \sigma^{\lambda\rho} \gamma^\nu \gamma^\mu \gamma^5)$$

$$+ \frac{i p_\nu}{m} \mathrm{tr.} (\sigma^{\lambda\rho} \sigma^{\nu\mu} \gamma^5) \Big\} [\bar{D}_\mu N]^i$$

+ similar term

$$= \frac{4i}{m} P_\nu \epsilon^{\lambda\rho\nu\mu} [\bar{D}_\mu N - \bar{N} D_\mu]^i \tag{6.37}$$

$$J^{i\rho 5}(DN) = -4i [\bar{D}^\rho N]^i$$

$$+ p_\nu p'_\lambda \mathrm{tr.} (-\gamma^\rho \gamma^5 \gamma^\lambda \gamma^5 \sigma^{\mu\nu}) [\bar{D}_\mu N]^i$$

+ similar terms

$$= -4i\,[\bar{D}^\rho N]^i$$

$$- \frac{p_\nu p'_\lambda}{m^2}\, 4i\,(g^{\rho\mu} g^{\lambda\nu} - g^{\rho\nu} g^{\mu\lambda})\,[\bar{D}_\mu N]^i$$

+ similar terms

$$= -4i\,[\bar{D}^\rho N]^i$$

$$-4i\,\frac{p_\nu p'^\nu}{m^2}[\bar{D}^\rho N]^i + 4i\,\frac{p^\rho p'^\mu}{m^2}[\bar{D}_\mu N]^i$$

+ similar terms

or

$$J^{i\rho 5} = -\frac{2iP^2}{m^2}\left[\bar{D}^\rho N - \bar{N} D^\rho\right]^i$$

$$+\frac{4i}{m^2}\left[p^\rho p'^\mu \bar{D}_\mu N - p'^\rho p^\mu \bar{N} D_\mu\right]^i \tag{6.38}$$

$$J^{i5} = p'_\lambda\, 4g^{\lambda\mu}[\bar{D}_\mu N]^i \qquad \text{+ similar term}$$

$$= 4Q^\mu[\bar{D}_\mu N - \bar{N} D_\mu]^i \tag{6.39}$$

where we have used $p^\mu \bar{D}_\mu = p'^\mu D_\mu = 0$.

Combining the results (6.26) to (6.39) to give J^{Ri} expanded according to (5.11), we have

$$J^i = \left[\frac{2P^2}{m^2}\{\bar{D}^\nu D_\nu\} + \frac{4}{m^2} Q^\nu \bar{D}_\nu Q^\nu D_\nu\right]^i$$

$$+ \frac{2P^2}{m^2}[\bar{N}N]^i_{3X+3Y} \tag{6.40}$$

$$J^{i\rho} = \Big[\frac{2P^2}{m^2}\bar{D}^\nu \gamma^\rho D_\nu + \frac{4}{m^2} Q^\nu \bar{D}_\nu \gamma^\rho Q^\nu D_\nu$$

$$-\frac{2}{m^2}\epsilon^{\rho\lambda\mu\nu} P_\lambda Q_\mu (\bar{N}D_\nu - \bar{D}_\nu N)\Big]^i$$

$$-\frac{2}{m^2}[\bar{N}\gamma^\rho N]^i_{5X-\gamma Y} + 4\frac{P^\rho}{m}[\bar{N}N]^i_{3X+3Y} \tag{6.41}$$

$$J^{i\lambda\rho} = \Big[\frac{2P^2}{m^2}\bar{D}^\nu \sigma^{\lambda\rho} D_\nu + \frac{4}{m^2} Q^\nu \bar{D}_\nu \sigma^{\lambda\rho} Q^\nu D_\nu$$

$$+ \frac{4i}{m}\epsilon^{\lambda\rho\mu\kappa} P_\kappa (\bar{N}D_\mu - \bar{D}_\mu N)\Big]^i$$

$$+ \frac{2P^2}{m^2}[\bar{N}\sigma^{\lambda\rho}N]^i_{5X-Y}$$

$$+ 4i(p^\rho p'^\lambda - p^\lambda p'^\rho)[\bar{N}N]^i_{-2X+4Y} \tag{6.42}$$

$$J^{i\rho 5} = \Big[\frac{P^2}{m^2}2\bar{D}^\nu i\gamma^\rho\gamma^5 D_\nu + \frac{4}{m^2} Q^\nu \bar{D}_\nu i\gamma^\rho\gamma^5 Q^\nu D_\nu$$

$$-\frac{2i}{m^2}P^2(\bar{N}D^\rho - \bar{D}^\rho N)$$

$$- +\frac{4i}{m^2}\{p'^\rho p^\nu \bar{N}D_\nu - p^\rho p'^\nu \bar{D}_\nu N\}\Big]^i$$

$$+ \frac{2P^2}{m^2}[\bar{N}i\gamma^\rho\gamma^5 N]^i_{5X-Y} \tag{6.43}$$

$$J^{i5} = \left[\frac{P^2}{m^2} 2\bar{D}^\mu \gamma^5 D_\mu + \frac{4}{m^2} Q^\mu \bar{D}_\mu \gamma^5 Q^\nu D_\nu + 4\frac{Q^\mu}{m}(\bar{D}_\mu N - \bar{N} D_\mu)\right]^i + \frac{2P^2}{m^2}\left[\bar{N}\gamma^5 N\right]^i_{5\gamma-\gamma} \qquad (6.44)$$

The baryon meson vertex $\bar{\Psi}^{ABC}\ \psi_{A'BC}\ \phi^{A'}_{A}$ can also be written in the form

$$J^{Ri}\,\phi_R{}^i \qquad (6.45)$$

However in view of equations (5.61) and (5.63), i.e.

$$\phi_{\mu\nu} = \frac{-i}{\mu}\left(Q_\mu \phi_\nu - Q_\nu \phi_\mu\right) \qquad (6.46)$$

and

$$\phi_{\mu 5} = \frac{-i}{\mu} Q_\mu \phi_5 \qquad (6.47)$$

(6.45) further simplifies to

$$\phi^i_5\,\tilde{J}^{i5} + \phi^i_\mu\,\tilde{J}^{i\mu} \qquad (6.48)$$

where

$$\tilde{J}^{i5} = J^{i5} + \frac{Q_\rho}{i\mu} J^{i\rho 5} \qquad (6.49)$$

and

$$\tilde{J}^{i\rho} = J^{i\rho} + \frac{Q_\lambda}{i\mu} J^{i\lambda\rho} \qquad (6.50)$$

Explicite expressions for $\tilde{J}^{i5}$ and $\tilde{J}^{i\rho}$ are

$$\tilde{J}^{i5} = \left(1 + \frac{2m}{\mu}\right) J^{i5} \qquad (6.51)$$

and

$$\tilde{J}^{i\rho} = \Big[\frac{2P^2}{m^2} \bar{D}^\nu \Big\{ \Big(1+\frac{2m}{N}\Big)\gamma^\rho - \frac{P^\rho}{N} \Big\} D_\nu$$

$$+ \frac{4}{m^2} Q^\nu \bar{D}_\nu \Big\{ \Big(1+\frac{2m}{N}\Big)\gamma^\rho - \frac{P^\rho}{N} \Big\} Q^\nu D_\nu$$

$$- \frac{4}{m^2}\Big(1+\frac{2m}{N}\Big) \epsilon^{\rho\kappa\lambda\nu} P_\kappa Q_\lambda \big(\bar{N} D_\nu - \bar{D}_\nu N\big) \Big]^i$$

$$+ \frac{4P^\rho}{m}\Big(1+\frac{Q^2}{2Nm}\Big) [\bar{N} N]^i_{3x+3Y}$$

$$- \frac{2}{m^2}\Big(1+\frac{2m}{N}\Big) [\bar{N} \gamma^\rho N]^i_{5x-Y} \qquad (6.52)$$

All these expressions have been derived under the assumption that the particles described by the fields $D_{\alpha\beta\gamma}$ and $N_{[\alpha\beta]\gamma}$ have the same mass m and that the vector and scalar particles described by $\bar{\Phi}^{B}_{A}$ have the same mass μ. To obtain the corresponding expressions when this assumption is dropped as discussed at the beginning of this chapter, the following prescription can be applied: terms involving $\bar{D}_\mu$ and D_μ should have their momentum dependent coefficients changed thus, $p^\rho/m \rightarrow p^\rho/m_D$, $p'/m \rightarrow p'/m_D$, where m_D is the new mass associated with $D_{\alpha\beta\gamma}$; similarly terms involving $\bar{N}$ and N simply change $m \rightarrow m_N$; terms involving $\bar{D}_\mu$ and N should be changed appropriately thus $p^\rho/m \rightarrow p^\rho/m_D + p'^\rho/m_N$.

This prescription is straightforward for the expressions (6.40) to (6.44). Equations (6.49) and (6.50) should have $\mu \rightarrow \mu^{(1)}$ and $\mu \rightarrow \mu^{(2)}$ respectively.

Equations (6.51) and (6.52) can then be modified by the changes

$$\mu \rightarrow \mu^{(1)} \quad \text{in} \quad (6.51)$$
$$\mu \rightarrow \mu^{(2)} \quad \text{in} \quad (6.52)$$

the factor $\frac{2m}{\mu} \rightarrow \frac{2m_D}{\mu^{(i)}}$, $\frac{2m_N}{\mu^{(i)}}$, $\frac{m_D+m_N}{\mu^{(i)}}$

according as it multiplies a term involving $(\overline{D}_\mu, D_\mu)$, $(\overline{N}, N)$ or $(\overline{D}_\mu, N)$; otherwise the previous rules are sufficient.

We are now in a position to consider the coupling of the electromagnetic field to the baryon structure. Following Delbourgo et al.[3] we suppose that the photon couples to the baryon through the meson cloud; the coupling of the photon to the vector-meson being given by the effective gauge-invariant interaction

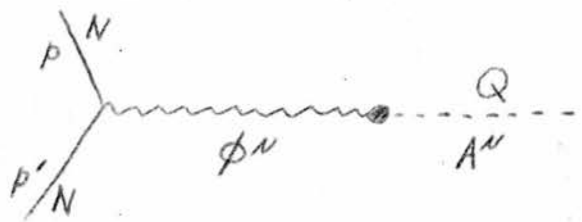
$$f(Q^2)\, F_{\mu\nu}\, \Phi^{(U)\mu\nu} \tag{6.53}$$

where

$$\Phi^{(U)\mu\nu} = \phi^{3\mu\nu} + 3^{-1/2}\, \phi^{8\mu\nu}$$

is the appropriate combination of charged vector mesons.

Considering the second order vertex



(6.54)

we are lead to the expressions

$$J_e^\rho = \frac{P^\rho}{2m}\left(1 + \frac{Q^2}{2\mu m}\right) F(Q^2) \left[\overline{N} N\right]^{(U)}_F \tag{6.55}$$

$$J_M^\rho = -\left(1 + \frac{2m}{\mu}\right) F(Q^2) \left[\overline{N}\, \frac{r^\rho}{4m^2}\, N\right]^{(U)}_{D+\frac{2}{3}F}$$

where

$$F = X + Y \quad ; \quad D = X - Y \,.$$

for the electric and magnetic form factors where

$$F(Q^2) \propto \frac{f(Q^2)}{Q^2 - \mu^2}$$

In order to establish (6.55) we introduce creation and annihilation operators for the particles of (6.54).

Thus

$$\begin{aligned} A_\mu(x) &= \sum_k \left(A_\mu^\dagger(k) e^{ikx} + A_\mu(k) e^{-ikx} \right) \\ N(x) &= \sum_k \left(U_k a_k e^{ikx} + V_k b_k^\dagger e^{-ikx} \right) \\ \bar{N}(x) &= \sum_k \left(\bar{U}_k a_k^\dagger e^{-ikx} + \bar{V}_k b_k e^{ikx} \right) \end{aligned} \qquad (6.56)$$

where we have dropped the isospin suffixes. We wish to consider an expression of the type f(p', p, Q)

$$\begin{aligned} &f(p', p, Q) \\ &= \langle 0 | a_{p'} \int dx\, dx' \, T \left\{ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) f(Q^2) \right. \\ &\qquad \left. \times \bar{\psi}' W_\lambda \psi' \phi'^\lambda \right\} a_p^\dagger A_\sigma^\dagger(Q) | 0 \rangle \end{aligned} \qquad (6.57)$$

where W_λ denotes an operator expression such as

$$\frac{p_\lambda}{2m}\left(1 + \frac{Q^2}{2\mu m}\right) \quad \text{or} \quad \left(1 + \frac{2m}{\mu}\right) \frac{k_\lambda}{4m^2} \quad .$$

Fourier transforming (6.57) with the aid of the expansions (6.56) and contracting the virtual vector meson fields we are lead to

$$\sum_{\kappa k'} \langle 0 | a_{p'} a^\dagger_{k'} a_{k'-\kappa} a^\dagger_p | 0 \rangle$$
$$\times \; \langle 0 | A_\nu(\kappa) A^\dagger_\sigma(Q) | 0 \rangle$$
$$\times \; \bar{U}_{k'} U_{k'-\kappa} \left\{ \kappa^2 \Delta^{\nu\lambda}(\kappa) + \kappa_\mu \kappa^\nu \Delta^{\mu\lambda}(\kappa) \right\} W_\lambda \tag{6.58}$$

where $\Delta^{\mu\nu}(\kappa)$ is the propagator for the vector meson

$$\Delta^{\mu\nu}(\kappa) = \left(g^{\mu\nu} - \frac{\kappa^\mu \kappa^\nu}{\mu^2} \right) \frac{1}{\kappa^2 - \mu^2} \tag{6.59}$$

Introducing the gauge condition $\kappa_\nu A^\nu = 0$ we see that the second term in the curly brackets of (6.58) drops out and, using the expression (6.59), that $\Delta^{\mu\nu}$ may be replaced by $\frac{g^{\mu\nu}}{\kappa^2 - \mu^2}$

Noting also that the summand of (6.58) contributes only where $p' + k' - \kappa = k' + p$ and $\kappa = Q$ we are left with

$$\sum_{k'} \langle 0 | a_{p'} a^\dagger_{k'} a_{k'-Q} a^\dagger_p | 0 \rangle$$
$$\times \; \langle 0 | A_\nu(Q) A^\dagger_\sigma(Q) | 0 \rangle$$
$$\times \; \bar{U}_{k'} U_{k'-Q} \frac{Q^2}{Q^2 - \mu^2} W^\nu$$

which is equivalent to (6.55); as required.

From (6.55) we have the result

$$\mu_p = \left(1. + \frac{2m}{\mu} \right)$$
$$\mu_n = -\frac{2}{3} \left(1 + \frac{2m}{\mu} \right) \qquad \text{(Bohr magnetons)} \tag{6.60}$$

In (6.60) m and μ represent the mean mass of the $\widetilde{U}(8)$ multiplets, but the same form, (6.60), is obtained if spin dependent mass splitting is allowed. In this case m and μ are the masses of the nucleon and ρ -meson respectively.

The result (6.60) was first obtained by Delbourgo et al.[3] from $\widetilde{U}(12)$ theory. For $\widetilde{U}(12)$ one introduces F and D type couplings as was done for SU(6).

DISCUSSION.

In Chapter 6 we have shown how the magnetic moments of the neutron and proton may be obtained from $\widetilde{U}(8)$ theory. It should be pointed out that the coupling of particles of the $\widetilde{U}(8)$ multiplets is non-local. There are a few generalisations of the couplings which we have introduced called 'exceptional' couplings. These are obtained by the introduction of the operators $(\not{P})_A^{A'}$ or $(\not{Q})_A^{A'}$ between the various summed suffixes of the interaction. In the non-relativistic limit such modifications are the same as a non-exceptional couplings or simply vanish. We should not then expect any changes to the prediction of magnetic moments by changes of this kind.

It has been found that $\widetilde{U}(12)$ theory is not consistent with the simultaneous requirement of Unitarity and Crossing even when exceptional coupling modifications are allowed[20].

These problems have been discussed by F. Gürsey[21] who shows how to construct an SU(6) group for particles of arbitrary momentum.

In the future we can anticipate the continuing development of symmetry theories. Many of the observed resonances do not have a certain place in present schemes and new resonances are still being discovered.

APPENDIX A

The Gamma-Matrices

The γ-matrices are defined by the relation

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \tag{A.1}$$

where we choose for our metric, $g_{\mu\nu} = g_{\mu\mu}\delta_{\mu\nu}$

with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$;

the indices μ, ν taking on the values 0, 1, 2, 3.

$g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ may be used to raise and lower suffixes thus:

$$\gamma^\mu = g^{\mu\nu}\gamma_\nu \tag{A.2}$$

Under a Lorentz transformation $a^\mu{}_\nu$ the matrices

$$\gamma'^\mu = a^\mu_\nu \gamma^\nu \tag{A.3}$$

satisfy

$$\gamma'^\mu = S^{-1}\gamma^\mu S \tag{A.4}$$

where, if $a^\mu{}_\nu$ is an infinitesimal transformation, i.e.

$$a^\mu_\nu = g^\mu_\nu + \alpha^\mu_\nu \tag{A.5}$$

with $\alpha^\mu{}_\nu$ infinitesimal, S has the form

$$S = 1 + \frac{i}{2}\alpha_{\mu\nu}\sigma^{\mu\nu} \tag{A.6}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{A.7}$$

For finite transformations S is the corresponding exponential.

The invariance of the Dirac equation

$$(\gamma^{\mu} \partial_{\mu} + m)\psi(x) = 0 \tag{A.8}$$

i.e.

$$(\gamma^{\nu}\partial'_{\nu} + m)\psi'(x') = 0 \tag{A.9}$$

is guaranteed by the requirement

$$\psi'(x') = S\psi(x) \tag{A.10}$$

since the hermitian adjoint spinor satisfies

$$\psi'(x')^{+} = \psi(x)^{+} S^{+} \tag{A.11}$$

Thus if we can find a matrix A such that

$$S^{+} A S = A, \tag{A.12}$$

we can form an invariant

$$\bar{\psi}\,\psi$$

where the adjoint spinor $\bar{\psi}$ is defined by

$$\bar{\psi} = \psi^{+} A \tag{A.13}$$

Equation (A.12) will be satisfied if A satisfies

$$\gamma^{\mu +} = A\gamma^{\mu} A^{-1}, \quad A^{+} = A \tag{A.14}$$

Note: $\gamma^{\mu\,+}$ is a representation of the γ-matrices as can easily be checked by forming the adjoint of (A.1).

A particular representation for the γ-matrices in terms of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.15}$$

is given by

$$\gamma_\ell = i\begin{pmatrix} 0 & \sigma_\ell \\ \sigma_\ell & 0 \end{pmatrix} \ (\ell = 1,2,3), \qquad \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{A.16}$$

In this representation γ_0 has the properties required of A.

We note also that in this representation

$$\begin{aligned} \gamma^+_\ell &= -\gamma_\ell \quad (\ell = 1,2,3) \\ \gamma^+_0 &= \gamma_0 \end{aligned} \tag{A.17}$$

For any representation satisfying (A.17) we may choose $A = \gamma_0$.

The matrices γ_μ are a subset of all 4 x 4 matrices. We may form 16 independent matrices Γ^R $R = 0,1,2,\ldots\ldots, 15$ by forming products thus:

$$\Gamma^R = I, \gamma^\mu, \sigma^{\mu\nu}, i\gamma^\mu\gamma^5, \gamma^5 (= \gamma^0\gamma^1\gamma^2\gamma^3) \tag{A.18}$$

The phases in (A.18) have been chosen so that

$$\Gamma^{R\dagger} = A \Gamma^R A^{-1} \tag{A.19}$$

It follows that $\bar\psi\psi$ is invariant under the extended group of transformations $\mathcal{S}$ which infinitesimally are given by

$$\mathcal{S}' = 1 + i\varepsilon_R \Gamma^R \tag{A.20}$$

ε_R real

$\bar\psi\psi$ is the only invariant under this group.

The transformations $\mathcal{S}$ form the group $\tilde{U}(4)$. Under the transformations S there is a second invariant we can form between ψ and ψ^T.

To see this we note that $\gamma_\mu{}^T$ and $\gamma_\mu{}^*$ satisfy (A.1) and also $-\gamma_\mu$, $-\gamma_\mu{}^*$, $-\gamma_\mu{}^T$, $-\gamma_\mu{}^\dagger$.

We look for matrices B and C such that

$$\gamma^{\mu\,T} = -C\gamma^\mu C^{-1} \quad ; \quad C^T = -C \tag{A.21}$$

$$\gamma^{\mu *} = -B\gamma^\mu B^{-1} \quad ; \quad B^* = B \tag{A.22}$$

In the particular representation (A.16) the matrix γ_2 has the properties of C, while B may be taken as the matrix

$$\gamma_2\gamma_0 \quad .$$

We note in general that

$$-\gamma^\mu = \gamma^5\gamma^\mu\gamma^{5\,-1} \tag{A.23}$$

γ^5 has the properties

$$\gamma^{5\,2} = -I \quad ; \quad \{\gamma^5, \gamma^\mu\} = 0 \tag{A.24}$$

Equation (A.21) implies

$$\sigma^{\mu\nu\,T} = -C\sigma^{\mu\nu}C^{-1} \tag{A.25}$$

$\psi^T C \psi$ is invariant for

$$\psi'^T(x')\,C\,\psi'(x') = \psi(x)^T S^T C\, S\,\psi(x) = \psi^T(x)\,C\,\psi(x)$$

i.e. if $S^T C S = C$ (A.26)

and this follows from (A.6) and (A.25).

It follows from (A.21) that C transforms according to

$$C' = (S^{+1})^T C (S^{+1}) \tag{A.27}$$

We introduce the following suffix convention: a tensor field $\psi_{\alpha\beta\ldots}^{\gamma\delta\ldots}$ with upper and lower suffixes has the property that for each lower suffix it transforms under S and for each upper suffix it transforms under S^{-1}, thus

$$\psi'^{\gamma\delta\ldots}_{\alpha\beta\ldots} = S_\alpha^{\alpha'} S_\beta^{\beta'} \ldots (S^{-1})^\gamma_{\gamma'} (S^{-1})^\delta_{\delta'} \ldots \psi^{\gamma'\delta'\ldots}_{\alpha'\beta'\ldots} \tag{A.28}$$

The matrices Γ^R thus have one upper suffix and one lower one. The matrix C has two upper indices.

The matrix A and the hermitian adjoint field ψ^+ do not fit into this classification and it is always necessary to multiply ψ^+ by A before forming invariants. It is convenient to always work with $\overline{\psi}$; A is not then required.

In forming invariant quantities we only have to be sure that summations over spinor indices are between upper and lower cases.

We mention that C^{-1} has two lower suffixes.

It is sometimes convenient when working with fields with more than one suffix to expand them in terms of γ-matrices or matrices of the form $\Gamma^R C^{-1}$. Thus we may expand fields ϕ_α^β and $\psi_{\alpha\beta\gamma}$

$$\phi_\alpha^\beta = \phi_R (\Gamma^R)_\alpha^\beta \tag{A.28}$$

or

$$\psi_{\alpha\beta\gamma} = \psi_{\alpha R} (\Gamma^R C^{-1})_{\beta\gamma} \tag{A.29}$$

We may wish to know the corresponding expansions of ϕ_β^α and

$\bar{\psi}^{\alpha\beta\gamma}$ in terms of $\bar{\phi}_R$ and $\bar{\psi}^{\alpha}_{R}$, where, in analogy with Dirac spinors we mean by $\bar{\phi}^{\alpha}_{\beta}$, $\bar{\psi}^{\alpha\beta\gamma}$ the tensors

$$\bar{\phi}^{\alpha}_{\beta} = (A^{-1})_{\beta\beta'} (\phi^{+})^{\beta'}_{\alpha'} A^{\alpha'\alpha} \tag{A.30}$$

$$\bar{\psi}^{\alpha\beta\gamma} = \psi^{+}_{\alpha'\beta'\gamma'} A^{\alpha'\alpha} A^{\beta'\beta} A^{\gamma'\gamma} \tag{A.31}$$

Putting

$$\bar{\phi}^{\alpha}_{\beta} = \phi_R (\tilde{\Gamma}^R)^{\alpha}_{\beta}$$
$$\bar{\psi}^{\alpha\beta\gamma} = \bar{\psi}^{\alpha}_{R} (X^R)^{\beta\gamma}$$

We have

$$(\tilde{\Gamma}^R)^{\alpha}_{\beta} = (A^{-1})_{\beta\beta'} (\Gamma^{R*})^{\beta'}_{\alpha'} A^{\alpha'\alpha}$$

$$(X^R)^{\beta\gamma} = (\Gamma^{R*} C^{-1*})_{\beta'\gamma'} A^{\beta'\beta} A^{\gamma'\gamma}$$

The suffixes summed over in the above expressions do not transform under S and S^{-1} but under S^{+} and $(S^{+})^{-1}$. The remaining suffixes however are consistent with our convention. Taking into account the positioning of the unsummed suffixes, it follows that

$$(\tilde{\Gamma}^R)^{\alpha}_{\beta} = (\Gamma^R)^{\alpha}_{\beta}$$

and

$$(X^R)^{\beta\gamma} = (C\,\Gamma^R)^{\beta\gamma} \quad .$$

Note in particular that corresponding to $(C^{-1})_{\alpha\beta}$ we have $C^{\alpha\beta}$. These results may be worked out using the particular representation (A.16) if one follows through carefully the transformation properties. This is necessary because, for instance, numerically $C^{\alpha\beta} = -C^{-1}_{\alpha\beta}$ in this representation.

The adjoint equations corresponding to (A.28) and (A.29) are thus

$$\bar{\phi}^{\alpha}_{\beta} = \phi_R (\Gamma^R)^{\alpha}_{\beta} \tag{A.32}$$

$$\bar{\Psi}^{\alpha\beta\gamma} = \bar{\Psi}^{\alpha}{}_R (C\Gamma^R)^{\beta\gamma} \tag{A.33}$$

Note incidentally that

$$\bar{\phi}^{\beta}_{\alpha} = \phi^{\beta}_{\alpha} \tag{A.34}$$

The matrices $(\Gamma^R C^{-1})_{\alpha\beta}$ split into two symmetry classes

$$C^{-1}_{\alpha\beta}, \quad (i\gamma^{\mu}\gamma^5 C^{-1})_{\alpha\beta}, \quad (\gamma^5 C^{-1})_{\alpha\beta} \tag{A.35}$$

are antisymmetric while

$$(\gamma^{\mu} C^{-1})_{\alpha\beta} \quad \text{and} \quad (\sigma^{\mu\nu} C^{-1})_{\alpha\beta} \tag{A.36}$$

are symmetric.

These relations follow directly from (A.21).

The matrix C exists under $\tilde{U}(4)$ but it is not then form invariant. The largest subgroup of $\tilde{U}(4)$ for which C is form invariant is the group with generators γ^{μ} and $\sigma^{\mu\nu}$. We denote this $\tilde{Sp}(4)$ in analogy with the simplectic group in 4 dimensions Sp(4) which differs from $\tilde{Sp}(4)$ only in the hermitian property of its generators.

APPENDIX B.

Normalisation of Tensor Fields.

The completely symmetric tensor ψ_{ABC} is an operator field associated with 364 states.

These states are conveniently labelled $|X_A\rangle$, $|X_{AB}\rangle$ $(B \neq A)$ and $|X_{ABC}\rangle$ $(A < B < C)$ and comprise an orthonormal set. Associated with these vectors are the field operators X_A, X_{AB}, X_{ABC} which are the destruction operators for the state vectors above.

$$\text{i.e.}\quad X_A\,|X_A\rangle = X_{AB}\,|X_{AB}\rangle = X_{ABC}\,|X_{ABC}\rangle = |0\rangle \quad \text{(no summation)} \tag{B.1}$$

where $|0\rangle$ denotes the vacuum state.

We draw the correspondence

$$\psi_{AAA} = X_A \tag{B.2}$$

$$\psi_{AAB} = \psi_{ABA} = \psi_{BAA} = 1/\sqrt{3}\; X_{AB} \tag{B.3}$$

$$\psi_{ABC} = \psi_{BCA} = \ldots\ldots\ldots = 1/\sqrt{6}\; X_{ABC} \tag{B.4}$$

The normalisation factors 1, $1/\sqrt{3}$ and $1/\sqrt{6}$ ensure

$$\bar{\psi}^{ABC}\,\psi_{ABC} = \bar{X}^A X_A + \bar{X}^{AB} X_{AB} + \bar{X}^{ABC} X_{ABC}$$

Now consider the expansion of ψ_{ABC} with respect to the subgroup $U(3) \times \tilde{U}(4)$ of $\tilde{U}(12)$. We write

$$\psi_{\alpha p,\, q\beta,\, r\gamma} = D_{pqr,\,\alpha\beta\gamma}$$

$$+ \left[\epsilon_{pqs} N_{[\alpha\beta]\gamma,\,r}^{\;s} + \epsilon_{qrs} N_{[\beta\gamma]\alpha,\,p}^{\;s} + \epsilon_{rps} N_{[\alpha\gamma]\beta,\,q}^{\;s} \right] \tag{B.5}$$

where $D_{pqr,\alpha\beta\gamma}$ is completely symmetric in both $\alpha\beta\gamma$ and pqr; $V_{\alpha\beta\gamma}$ is antisymmetric in $\alpha\beta\gamma$, and $N_{[\alpha\beta]\gamma}$ is of symmetry type $(2, 1)$ in $\alpha\beta\gamma$.

We can consider the field operators $D_{pqr,\alpha\beta\gamma}$ etc. to be the direct product states of the U(3) and $\hat{U}(4)$ field operators. We see from the above that the 364 operators X_A, X_{AB}, X_{ABC} separate the 1728 (non independent) components of ψ_{ABC} into 364 sets. Similarly $D_{pqr,\alpha\beta\gamma}$ can be split into 120 easily distinguishable sets. We may express $V_{\alpha\beta\gamma}$ in terms of 4 fields $V\rho$, thus

$$V_{\alpha\beta\gamma} = \frac{1}{6}\,\varepsilon_{\alpha\beta\gamma\rho}\, V\rho \tag{B.6}$$

Introducing a convenient set of orthonormal states corresponding to $N_{[\alpha\beta]\gamma,\,r}^{\;\;s}$ is a little more tricky, owing to the more complicated type of symmetry. Even when a sensible correspondence has been made, the fact that 3 non independent terms comprise the bracketed term of equation (B.5) will lead to an overall error in the normalisation of this bracketed term. Let us write, therefore $N_{[\alpha\beta]\gamma,\,r}^{\;\;s}$ as a sum of terms

$$z\, N_{[\alpha\beta]\gamma}\,.\, M_r^s\ , \tag{B.7}$$

where $N_{[\alpha\beta]\gamma}$ and M_r^s are field operators in $\hat{U}(4)$ and U(3) space respectively, and z is a normalisation factor to be determined.

M_r^s will be a linear combination of eight fields X_r^s $(r \neq s)$, U and V. Thus

$$M_r^s = X_r^s \qquad (r \neq s) \tag{B.8}$$

and

$$
\begin{aligned}
M_1^{\,1} &= \tfrac{1}{\sqrt{2}}\, U + \tfrac{1}{\sqrt{6}}\, V \\
M_2^{\,2} &= -\tfrac{1}{\sqrt{2}}\, V + \tfrac{1}{\sqrt{6}}\, V \\
M_3^{\,3} &= -\tfrac{2}{\sqrt{6}}\, V
\end{aligned}
\tag{B.9}
$$

The asymmetry of (B.8) arises from the trace condition

$$M_r^{\,r} = 0 \tag{B.10}$$

From (B.8) and (B.9) we have

$$\overline{M}_s^{\,r}\, M_r^{\,s} = \overline{X}_s^{\,r}\, X_r^{\,s} + \overline{U}U + \overline{V}V \tag{B.11}$$

$N_{[\alpha\beta]\gamma}$ satisfies the symmetry conditions

$$N_{[\alpha\beta]\gamma} = -N_{[\beta\alpha]\gamma} \tag{B.12}$$

and

$$N_{[\alpha\beta]\gamma} + N_{[\beta\gamma]\alpha} + N_{[\gamma\alpha]\beta} = 0 \tag{B.13}$$

Equation (B.12) is closely related to (B.9).

We introduce field operators

$X_{\alpha\beta}$ $(\alpha \neq \beta)$, and

$U_{\alpha\beta\gamma}$ and $W_{\alpha\beta\gamma}$ with $(\alpha < \beta < \gamma)$.

We draw the correspondence

$$
\left.
\begin{aligned}
N_{[\alpha\beta]\gamma} &= \tfrac{1}{\sqrt{2}}\, U_{\alpha\beta\gamma} + \tfrac{1}{\sqrt{6}}\, W_{\alpha\beta\gamma} = -N_{[\beta\alpha]\gamma} \\
N_{[\beta\gamma]\alpha} &= -\tfrac{1}{\sqrt{2}}\, U_{\alpha\beta\gamma} + \tfrac{1}{\sqrt{6}}\, W_{\alpha\beta\gamma} = -N_{[\gamma\beta]\alpha} \\
N_{[\gamma\alpha]\beta} &= -\tfrac{2}{\sqrt{6}}\, W_{\alpha\beta\gamma} = -N_{[\alpha\gamma]\beta}
\end{aligned}
\right\}
\tag{B.14}
$$

$$N_{[\alpha\beta]\alpha} = -N_{[\beta\alpha]\alpha} = X_{\alpha\beta} \tag{B.15}$$

We note that this prescription gives

$$\overline{N}^{[\alpha\beta]\gamma} N_{[\alpha\beta]\gamma} = \overline{X}^{\alpha\beta} X_{\alpha\beta} + \overline{U}^{\alpha\beta\gamma} U_{\alpha\beta\gamma} + \overline{W}^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \tag{B.16}$$

In order to find the value of the normalisation factor z of equation (B.7) we consider the special case of (B.5) with

$\alpha \neq \beta, \; p \neq q$.

$$\Psi_{p\alpha,p\alpha,q\beta} = D_{ppq,\alpha\alpha\beta} + \left[\epsilon_{pqs} N_{[\alpha\beta]\alpha p}^{\;s} + \epsilon_{qps} N_{[\beta\alpha]\alpha p}^{\;s}\right]$$

$$\text{or } \Psi_{p\alpha,p\alpha,q\beta} = D_{ppq,\alpha\alpha\beta} + 2\epsilon_{pqs} N_{[\alpha\beta]\alpha p}^{\;s} \tag{B.17}$$

Multiplying (B.17) on the right by the adjoint expression, and forming the vacuum expectation value we have

$$\frac{1}{3} = \frac{1}{9} + 4 \langle 0 | N_{[\alpha\beta]\alpha p}^{\;s} \overline{N}^{[\alpha\beta]\alpha p}_{\;s} | 0 \rangle$$

or, putting $N_{[\alpha\beta]\alpha p}^{\;s} = z \; X_{\alpha\beta} \, X_p^s$

we have

$$z = 1/3 \times (2)^{\frac{1}{2}} \tag{B.18}$$

We have now to relate $D_{\alpha\beta\gamma}$, $N_{[\alpha\beta]\gamma}$ to the coefficients $D_{\mu\gamma}$, $D_{\mu\nu\gamma}$, K_γ, N_γ, $N_{\mu\gamma}$ in the expansions

$$D_{\alpha\beta\gamma} = (\gamma^\mu C^{-1})_{\alpha\beta} D_{\mu\gamma} + \tfrac{1}{2}(\sigma^{\mu\nu} C^{-1})_{\alpha\beta} D_{\mu\nu\gamma} \tag{B.19}$$

$$N_{[\alpha\beta]\gamma} = (C^{-1})_{\alpha\beta} K_\gamma + (\gamma^5 C^{-1})_{\alpha\beta} N_\gamma + (i\gamma^\mu\gamma^5 C^{-1})_{\alpha\beta} N_{\mu\gamma} \tag{B.20}$$

where we suppose $D_{\mu\nu} = - D_{\nu\mu}$.

Symmetry imposes a number of restraints

$$\gamma^\mu D_\mu = \sigma^{\mu\nu} D_{\mu\nu} = 0 \quad ; \quad D_\rho + i\gamma^\nu D_{\rho\nu} = 0 \tag{B.21}$$

and

$$K - \gamma^5 N - i\gamma^\mu\gamma^5 N_\mu = 0 \tag{B.22}$$

We are of course not really interested in all 364 fields of ψ_{ABC}. This is because after the application of the Bargmann-Wigner equations only 56 independent fields are possible. Let us consider the situation in the rest frame. The equations

$$(\not{p} - m)_\alpha^{\alpha'} D_{\alpha'\beta\gamma} = 0$$

and (B.23)

$$(\not{p} - m)_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma} = 0$$

become

$$(1 - \gamma_0)_\alpha^{\alpha'} D_{\alpha'\beta\gamma} = 0$$

and (B.24)

$$(1 - \gamma_0)_\alpha^{\alpha'} N_{[\alpha'\beta]\gamma} = 0$$

In the particular representation of the γ-matrices of appendix ,
this means that only those components for which $\alpha = 1, 2$ exist.
That is

$D_{\alpha\beta\gamma} = 0$ if any of $\alpha, \beta, \gamma = 3, 4$

$N_{\alpha\beta\gamma} = 0$ if any of $\alpha, \beta, \gamma = 3, 4$

but otherwise there is no restriction.

We have already found out the meaning of equation (B.23) or (B.24) in terms of relations between K_γ, N_γ, $N_{\mu\gamma}$ and $D_{\mu\nu\gamma}$, $D_{\mu\gamma}$, so we need only find out what N_γ and $D_{\mu\gamma}$ are in the present situation.

Thus combining (B.20) with (5.52) and (5.54) we have in the rest frame

$$N_{[\alpha\beta]\gamma} = (\gamma^5 C^{-1})_{\alpha\beta} N_\gamma + (\gamma^0\gamma^5 C^{-1})_{\alpha\beta} V_\gamma$$

or, in our particular γ-matrix representation

$$N_{[12]1} = 2 N_1$$

and

$$N_{[12]2} = 2N_2$$

Thus the physical operators in the rest frame are $X_{\alpha\beta}$ with $\alpha = 1$, $\beta = 2$ or $\alpha = 2$, $\beta = 1$.

These states correspond to

$$N_1 = \tfrac{1}{2} X_{12}$$

$$\text{and} \quad N_2 = -\tfrac{1}{2} X_{21} \quad \text{respectively.} \tag{B.25}$$

For $D_{\alpha\beta\gamma}$ we observe from (5.45) that $D_{\mu\alpha} = 0$ for $\alpha = 3,4$. From (5.47) $D_{o\alpha} = 0$, and from (5.21) $\gamma^i D_{i\alpha} = 0$. This leaves us with 6 fields $D_{i\alpha}$ ($\alpha = 1,2$; $i = 1,2,3$) and two conditions.

From equation (5.49) the non zero components of $D_{\mu\nu}$ are

$$D_{oi\alpha} = -D_{io\alpha} = -i D_{i\alpha} \qquad \alpha = 1,2$$

Combining these results with (B.19)

$$\begin{aligned} D_{\alpha\beta\gamma} &= (\gamma^i C^{-1})_{\alpha\beta} D_{i\gamma} + (i\sigma^{io} C^{-1})_{\alpha\beta} D_{i\gamma} \\ &= \left\{ \left(\gamma^i - \tfrac{1}{2}[\gamma^i, \gamma^0]\right) C^{-1} \right\}_{\alpha\beta} D_{i\gamma} \\ &= \left(\gamma^i (1-\gamma^0) C^{-1} \right)_{\alpha\beta} D_{i\gamma} \end{aligned} \tag{B.26}$$

$$D_{11\gamma} = -\beta i\, D_{1\gamma} - 2\, D_{2\gamma}$$

$$D_{22\gamma} = 2i\, D_{1\gamma} - 2\, D_{2\gamma}$$

$$D_{12\gamma} = 2i\, D_{3\gamma}$$

or

$$D_{1\gamma} = \frac{1}{4i}\left(D_{22\gamma} - D_{11\gamma}\right)$$

$$D_{3\gamma} = \frac{1}{2i} D_{12\gamma}$$

$$D_{2\gamma} = -\frac{1}{4}\left(D_{11\gamma} + D_{22\gamma}\right)$$

In terms of $\mathbb{X}_\alpha$, $\mathbb{X}_{\alpha\beta}$, $\mathbb{X}_{\alpha\beta\gamma}$ (a basis for $D_{\alpha\beta\gamma}$) we have

$$
\begin{aligned}
D_{1,1} &= \frac{1}{4i}\left(\frac{1}{\sqrt{3}}\mathbb{X}_{21} - \mathbb{X}_1\right) \\
D_{1,2} &= \frac{1}{4i}\left(\mathbb{X}_2 - \frac{1}{\sqrt{3}}\mathbb{X}_{12}\right) \\
D_{2,1} &= -\frac{1}{4}\left(\mathbb{X}_1 + \frac{1}{\sqrt{3}}\mathbb{X}_{21}\right) \\
D_{2,2} &= -\frac{1}{4}\left(\frac{1}{\sqrt{3}}\mathbb{X}_{12} + \mathbb{X}_2\right) \\
D_{3,1} &= \frac{-i}{2\sqrt{3}}\mathbb{X}_{12} \\
D_{3,2} &= \frac{-i}{2\sqrt{3}}\mathbb{X}_{21}
\end{aligned}
\qquad \text{(B.27)}
$$

From (B.25) we discover

$$\overline{N}^\gamma N_\gamma = \frac{1}{4}\left\{\overline{X}^{12} X_{12} + \overline{X}^{21} X_{21}\right\} \qquad \text{(B.28)}$$

and from (B.27)

$$\overline{D}^{\mu\gamma} D_{\mu\gamma} = \frac{1}{8}\left\{\overline{\mathbb{X}}^1 \mathbb{X}_1 + \overline{\mathbb{X}}^2 \mathbb{X}_2 + \overline{\mathbb{X}}^{12} \mathbb{X}_{12} + \overline{\mathbb{X}}^{21} \mathbb{X}_{21}\right\} \qquad \text{(B.29)}$$

From (B.28) and (B.29) we conclude that if $|d\rangle$ is a normalised state-vector belonging to the 40 states of the decuplet, and $|n\rangle$ is any of the 16 states belonging to the octoplet then

$$\langle d|\bar{D}^{\mu\gamma,pqr}D_{\mu\gamma,pqr}|d\rangle = \frac{1}{8} \tag{B.30}$$

and

$$\langle n|\bar{N}^{\gamma p}{}_{q}\, N_{\gamma p}{}^{q}|n\rangle = \frac{1}{72} \tag{B.31}$$

The states $|\Xi_1\rangle, |\Xi_{12}\rangle, |X_{21}\rangle, |X_2\rangle$, are eigenstates of the operator J_3 with eigenvalues $3/2$, ½, -½, $-3/2$ respectively. This follows from the fact that we have chosen a representation of the γ-matrices in which

$\sigma^{12} = \frac{i}{2}[\gamma^1, \gamma^2]$ is diagonal.

Similarly $|X_{12}\rangle$ and $|X_{21}\rangle$ are eigenstates of J_3 with eigenvalues ½ and -½ respectively.

We wish to consider now what differences there are between $\tilde{U}(12)$ and $\tilde{U}(8)$ with respect to normalisation.

In $\tilde{U}(8)$ we have instead of equation (B.5) the equation

$$\psi_{p\alpha,q\beta,r\gamma} = D_{pqr,\alpha\beta\gamma} + [\epsilon_{pq}N_{[\alpha\beta]\gamma,r} + \epsilon_{qr}N_{[\beta\gamma]\alpha,p} + \epsilon_{rp}N_{[\gamma\alpha]\beta q}] \tag{B.32}$$

The operators X_A, X_{AB}, X_{ABC}, $X_{\alpha\beta}$, $U_{\alpha\beta\gamma}$, $W_{\alpha\beta\gamma}$, Ξ_α, $\Xi_{\alpha\beta}$, $\Xi_{\alpha\beta\gamma}$, Ξ_p, Ξ_{pq}, are retained, but now A,B,C run from 1 to 8 only and p,q take the values 1 and 2 only. In place of (B.7) we express $N_{[\alpha\beta]\gamma,}$ as a sum of terms

$$Z_2\, N_{[\alpha\beta]\gamma}\, M_r \tag{B.33}$$

where in place of (B.8) and (B.9) we have simply

$$M_1 = U_1$$
$$M_2 = U_2 \qquad (B.34)$$

To evaluate Z_2 we consider the special case of (B.32) with $p \neq q$, $\alpha \neq \beta$

$$\psi_{p\alpha, p\alpha, q\beta} = D_{ppq, \alpha\alpha\beta} + 2\epsilon_{pq} N_{[\alpha\beta]\alpha, p} \qquad (B.35)$$

Multiplying (B.35) on the right with the adjoint expression, and forming the vacuum expectation value

$$\frac{1}{3} = \frac{1}{9} + 4 \langle 0 | \bar{N}^{[\alpha\beta]\alpha p} N_{[\alpha\beta]\alpha p} | 0 \rangle$$

or, putting $N_{\alpha\beta\ \alpha p} = Z_2 X_{\alpha\beta} U_p$,

$$Z_2 = 1/3\sqrt{2} \qquad (B.36)$$

Since the remainder of the analysis of U(12) normalisation involves only U(4) we will have in place of (B.30) and (B.31) the equations

$$\langle d | \bar{D}^{\mu\gamma, pqr} D_{\mu\gamma, pqr} | d \rangle = \frac{1}{8} \qquad (B.37)$$

and

$$\langle n | \bar{N}^{\gamma, r} N_{\gamma, r} | n \rangle = \frac{1}{72} \qquad (B.38)$$

We have so far investigated the implication of the Bargmann-Wigner equations in the rest frame. Even here we have used a particular representation of the γ-matrices. A simplifying property of these γ-matrices is the form of γ_o which makes the operator $1 - \gamma_o$

a diagonalprojection operator. A second useful property is that σ^{12} is diagonal. If we used a different set of γ-matrices or did not work in the rest frame the situation would be complicated by the fact that the field operators $X_{\alpha\beta}$, $U_{\alpha\beta\gamma}$, $V_{\alpha\beta\gamma}$, $\underline{X}_{\alpha}$, $\underline{X}_{\alpha\beta}$ $\underline{X}_{\alpha\beta\gamma}$ would cease to be a convenient set of operators, once the symmetry is reduced.

When not in the rest frame the operator $(\not p - m)$ replaces $m(\gamma_0 - 1)$. However, we have

$\not p = m\,\gamma_0'$ where γ_0' is the matrix obtained from the γ_μ by the Lorentz transformation $a_\mu{}^\nu$ from the rest frame to the frame in which the particle is at rest. i.e.

$$\gamma_\mu' = a_\mu{}^\nu \gamma_\nu \tag{B.39}$$

Under these circumstances we also have the relation

$$(\gamma_\mu')_\alpha^\beta = S_\alpha^{\alpha'} (\gamma_\mu)_{\alpha'}^{\beta'} (S^{-1})_{\beta'}^\beta \tag{B.40}$$

where $(S)_\alpha^\beta$ is the spinor transformation associated with $a_\mu{}^\nu$.

Thus for example the equation

$(\not p - m)_\alpha^{\alpha'}\ D_{\alpha'\beta\gamma} = 0$ is equivalent to $m(\gamma_0 - I)_\alpha^{\alpha'}\ \tilde{D}_{\alpha'\beta\gamma} = 0$

where

$$\tilde{D}_{\alpha\beta\gamma} = S_\alpha^{\alpha'}\ S_\beta^{\beta'}\ S_\gamma^{\gamma'}\ D_{\alpha'\beta'\gamma'} \tag{B.41}$$

A convenient set of operators to describe things with is evidently $\tilde{\underline{X}}_\alpha, \tilde{\underline{X}}_{\alpha\beta}, \tilde{\underline{X}}_{\alpha\beta\gamma}$, with

$$\tilde{\underline{X}}_\alpha = \tilde{D}_{\alpha\alpha\alpha}$$

$$\frac{1}{\sqrt{3}}\tilde{\underline{X}}_{\alpha\beta} = \tilde{D}_{\alpha\alpha\beta} \qquad \alpha \neq \beta$$

$$\frac{1}{\sqrt{6}}\tilde{\underline{X}}_{\alpha\beta\gamma} = \tilde{D}_{\alpha\beta\gamma} \qquad \alpha < \beta < \gamma$$

and similarly for other spinor fields.

APPENDIX C

Commutator and Anticommutator Relations.

The matrices Γ^R of the Dirac algebra, $\Gamma^R = I$, $\gamma_\mu, \sigma^{\mu\nu}$ $(= \frac{1}{2}[\gamma^\mu, \gamma^\nu])$, $\sigma^{\mu 5}$ $(= i\gamma^\mu \gamma^5)$, γ^5, satisfy

$$[I, \Gamma^R] = 0 \tag{C.1}$$

$$\{I, \Gamma^R\} = 2\Gamma^R \tag{C.2}$$

$$[\gamma^\lambda, \sigma^{\mu\nu}] = 2i(g^{\lambda\mu}\gamma^\nu - g^{\lambda\nu}\gamma^\mu) \tag{C.3}$$

$$\{\gamma^\lambda, {}^{\mu\nu}\} = -2\,\varepsilon^{\lambda\mu\nu\rho}\sigma_{\rho 5} \tag{C.4}$$

$$[\sigma^{\varkappa\lambda}, \sigma^{\mu\nu}] = 2i(g^{\varkappa\nu}\sigma^{\lambda\mu} + g^{\lambda\mu}\sigma^{\varkappa\nu} - g^{\varkappa\mu}\sigma^{\lambda\nu} - g^{\lambda\mu}\sigma^{\varkappa\nu}) \tag{C.5}$$

$$\{\sigma^{\varkappa\lambda}, \sigma^{\mu\nu}\} = 2(g^{\varkappa\mu}g^{\lambda\nu} - g^{\lambda\mu}g^{\varkappa\nu}) - 2\varepsilon^{\varkappa\lambda\mu\nu}\gamma^5 \tag{C.6}$$

$$[\sigma^{\lambda 5}, \sigma^{\mu\nu}] = 2i(g^{\lambda\mu}\sigma^{\nu 5} - g^{\lambda\nu}\sigma^{\mu 5}) \tag{C.7}$$

$$\{\sigma^{\lambda 5}, \sigma^{\mu\nu}\} = -2\,\varepsilon^{\lambda\mu\nu\rho}\gamma_\rho \tag{C.8}$$

$$[\gamma^5, \sigma^{\mu\nu}] = 0 \;;\; \{\gamma^5, \sigma^{\mu\nu}\} = \epsilon^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} \tag{C.9}$$

Here $\gamma^5 = \gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ (C.10)

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